STAR OPERATIONS FOR AFFINE HECKE ALGEBRAS

DAN BARBASCH AND DAN CIUBOTARU

To Wilfried Schmid with admiration

Abstract. In this paper, we consider the star operations for (graded) affine Hecke algebras which preserve certain natural filtrations. We show that, up to inner conjugation, there are only two such star operations for the graded Hecke algebra: the first, denoted $\star$, corresponds to the usual star operation from reductive p-adic groups, and the second, denoted $\bullet$ can be regarded as the analogue of the compact star operation of a real group considered by [ALTV]. We explain how the star operation $\bullet$ appears naturally in the Iwahori-spherical setting of p-adic groups via the endomorphism algebras of Bernstein projectives. We also prove certain results about the signature of $\bullet$-invariant forms and, in particular, about $\bullet$-unitary simple modules.

Contents

1. Introduction

This work is motivated by the results about the unitary dual obtained in the case of real reductive groups by Adams, van Leeuwen, Trapa, and Vogan [ALTV] on the one hand, and on the other hand, in work and conjectures of Schmid and Vilonen [SV]. The ultimate goal is to obtain an algorithm for computing hermitian forms of irreducible modules in the case of reductive p-adic groups.

In this paper, we initiate the study of invariant hermitian forms for the (graded) affine Hecke algebras that appear in the theory of unipotent representations of reductive p-adic groups (Lusztig [Lu3]). There are two main parts to our paper which we explain next.

1.1. We introduce star operations (conjugate-linear involutive anti-automorphims) for the affine Hecke algebra $\mathcal{H}$ with unequal parameters which preserve natural filtrations of $\mathcal{H}$ (section 3) and classify them in the corresponding setting of the graded affine Hecke algebras defined by Lusztig [Lu3]. The classification problem can be viewed as an analogue of the problem of classifying the star operations for the enveloping algebra $U(\mathfrak{g})$ of a complex semisimple Lie algebra which preserve $\mathfrak{g}$. Proposition 3.4.3 says that essentially there are only two such star operations: $\star$ and $\bullet$, see Definitions 2.3.1 and 3.3.2.

The anti-automorphism $\star$ is known to correspond to the natural star operation of the Hecke algebra of a reductive p-adic group, i.e., $f^\star(g) = \overline{f(g^{-1})}$, see [BM1, BM2].

On the other hand, the anti-automorphism $\bullet$ is the Hecke algebra analogue of the "compact star operation" for $(\mathfrak{g}, K)$-module studied in [ALTV]. In section 2, we explain that $\bullet$ appears naturally in the study of Iwahori-Hecke algebras via the projective (non-admissible) modules defined by Bernstein [Be]. The operation $\bullet$ for affine Hecke algebras also arises naturally in work of Opdam [Op2].

1.2. We study the basic properties of the signature of $\bullet$-invariant hermitian forms for finite dimensional $\mathbb{H}$-modules. We explain that every irreducible spherical $\mathbb{H}$-module with real central character admits an (explicit) nondegenerate $\bullet$-invariant hermitian form, Proposition 5.1.2. This result is generalized in [BC4], where we prove that every simple $\mathbb{H}$-module with real central character admits a nondegenerate $\bullet$-invariant hermitian form, and, moreover, this form can be normalized canonically (at least when $\mathbb{H}$ is of geometric type in the sense of Lusztig) to be positive definite on the lowest W-types. These results can be thought of Hecke algebra analogue of the similar results about c-invariant forms of $(\mathfrak{g}, K)$-modules [ALTV]. The formulations of some of our results were inspired by the ongoing work of Schmid and Vilonen [SV] aimed at using geometric methods to study unitarity. For example, the relation between the $\star$- and $\bullet$-signature characters of the tempered modules with real central characters (realized in the cohomology of Springer fibers) in section 4.2 is motivated by their work.

Using the Dirac operator defined in [BCT], we also prove that the only $\bullet$-unitary spherical $\mathbb{H}$-modules, where $\mathbb{H}$ has equal parameters, are the ones whose parameters lie in the closure of the $\rho^\vee$-cone, Theorem 6.3.3. Similar ideas lead to showing that every simple $\mathbb{H}$-module with central character $\rho^\vee$ has nontrivial Dirac cohomology, and moreover their Dirac cohomology spaces are essentially the same, Corollary 6.4.4.

1.3. These results were presented in two talks given by Dan Barbasch in 2013. The first was at the TSIMF conference at Sanya in January, as part of a special session organized by W. Schmid and S. Miller, the second at the conference in honor of W. Schmid's 70^{th} birthday in June of 2013. He would like to thank the organizers of these conferences for providing the means for mathematical researchers to have a very productive exhange of ideas.

The first author was partially supported by NSF grants DMS-0967386, DMS-0901104 and an NSA-AMS grant. The second author was partially supported by NSF DMS-1302122 and NSA-AMS 111016.

2. Star operations: the affine Hecke algebra

2.1. **The affine Hecke algebra.** Let $\mathcal{R} = (X, R, X^\vee, R^\vee, \Pi)$ be a based root datum [Sp]. In particular, $X, X^\vee$ are lattices in perfect duality $\langle\ ,\ \rangle : X \times X^\vee \to \mathbb{Z}$, $R \subset X\backslash\{0\}$ and $R^\vee \subset X^\vee\backslash\{0\}$ are the (finite) sets of roots and coroots respectively, and $\Pi \subset R$ is a basis of simple roots. Let W be the finite Weyl group with set of generators $S = \{s_\alpha : \alpha \in \Pi\}$. Set $W^e = W \ltimes X$, the extended affine Weyl group, and $W^a = W \ltimes Q$, the affine Weyl group, where Q is the root lattice of R.

The set $R^a = R^\vee \times \mathbb{Z} \subset X^\vee \times \mathbb{Z}$ is the set of affine roots. A basis of simple affine roots is given by $\Pi^a = (\Pi^\vee \times \{0\}) \cup \{(\gamma^\vee, 1) : \gamma^\vee \in R^\vee \text{ minimal}\}$. For every affine root $\mathbf{a} = (\alpha^\vee, n)$, let $s_{\mathbf{a}} : X \to X$ denote the reflection $s_{\mathbf{a}}(x) = x - ((x, \alpha^\vee) + n)\alpha$. The affine Weyl group W^a has a set of generators $S^a = \{s_{\mathbf{a}} : \mathbf{a} \in \Pi^a\}$. Let $\ell : W^e \to \mathbb{Z}$ be the length function with respect to S^a.

Let $\mathbf{q}$ be an indeterminate and let

$$L : S^a \to \mathbb{Z}_{\geq 0}$$

be a W^a-invariant function.

Definition 2.1.1 (Iwahori presentation)**.** The affine Hecke algebra $\mathcal{H}(\mathbf{q}) = \mathcal{H}(\mathcal{R}, \mathbf{q}, L)$ associated to the root datum $\mathcal{R}$ and parameters L is the unique associative, unital $\mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$-algebra with basis $\{T_w : w \in W^e\}$ and relations

(i) $T_w T_{w'} = T_{ww'}$, for all $w, w' \in W^e$ such that $\ell(ww') = \ell(w) + \ell(w')$;
(ii) $(T_s - \mathbf{q}^{2L(s)})(T_s + 1) = 0$ for all $s \in S^a$.

2.2. **The Bernstein presentation.** The affine Hecke algebra admits a second presentation due to Bernstein and Lusztig, see [Lu1].

A parameter set for $\mathcal{R}$ is a pair of functions (λ, λ^*),

$$\lambda : \Pi \to \mathbb{Z}_{\geq 0}, \quad \lambda^* : \{\alpha \in \Pi : \alpha^\vee \in 2X^\vee\} \to \mathbb{Z}_{\geq 0},$$

such that $\lambda(\alpha) = \lambda(\alpha')$ and $\lambda^*(\alpha) = \lambda^*(\alpha')$ whenever α, α' are $W-$conjugate. The relation with the parameters in the Iwahori presentation is:

$$\lambda(\alpha) = L(s_\alpha),\ \alpha \in \Pi, \quad \lambda^*(\alpha) = L(\hat{s}_\alpha),\ \alpha \in \Pi, \alpha^\vee \in 2X^\vee, \tag{2.2.1}$$

where $\hat{}$ is the unique nontrivial automorphism of the Dynkin diagram of affine type $\widetilde{C}_r$.

Definition 2.2.1 (Bernstein presentation)**.** The affine Hecke algebra $\mathcal{H}(\mathbf{q}) = \mathcal{H}^{\lambda,\lambda^*}(\mathcal{R}, \mathbf{q})$ associated to the root datum $\mathcal{R}$ with parameter set (λ, λ^*), is the associative algebra over $\mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$ with unit, defined by generators T_w, $w \in W$, and θ_x, $x \in X$ with relations:

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - \mathbf{q}^{2\lambda(\alpha)}) = 0, \text{ for all } \alpha \in \Pi, \tag{2.2.2}$$

$$T_w T_{w'} = T_{ww'}, \text{ for all } w, w' \in W \text{ such that } \ell(ww') = \ell(w) + \ell(w'),$$

$$\theta_x \theta_{x'} = \theta_{x+x'}, \text{ for all } x, x' \in X, \tag{2.2.3}$$

$$\theta_x T_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha(x)} = (\theta_x - \theta_{s_\alpha(x)})(\mathcal{G}(\alpha) - 1), \text{ where } x \in X, \alpha \in \Pi, \text{ and}$$

$$\mathcal{G}(\alpha) = \begin{cases} \frac{\theta_\alpha \mathbf{q}^{2\lambda(\alpha)} - 1}{\theta_\alpha - 1}, & \text{if } \alpha^\vee \notin 2X^\vee, \\ \frac{(\theta_\alpha \mathbf{q}^{\lambda(\alpha)+\lambda^*(\alpha)} - 1)(\theta_\alpha \mathbf{q}^{\lambda(\alpha)-\lambda^*(\alpha)} + 1)}{\theta_{2\alpha} - 1}, & \text{if } \alpha^\vee \in 2X^\vee. \end{cases} \tag{2.2.4}$$

We refer to [Lu1, section 3] for more details about the relations between the two presentations.

2.3. **Star operations.** There are two known star operations on $\mathcal{H}(\mathbf{q})$, i.e., conjugate linear involutive anti-automorphisms, $\star$ and $\bullet$.

Definition 2.3.1. (1) In the Iwahori presentation, $\star$ is defined on generators by

$$\mathbf{q}^\star = \mathbf{q}, \quad T_w^\star = T_{w^{-1}},\ w \in W^e. \tag{2.3.1}$$

In the Bernstein presentation, the equivalent definition is [BM2, section 5]:

$$\mathbf{q}^\star = \mathbf{q}, \quad T_w^\star = T_{w^{-1}},\ w \in W, \quad \theta_x^\star = T_{w_0} \cdot \theta_{-w_0(x)} \cdot T_{w_0}^{-1},\ x \in X, \tag{2.3.2}$$

where w_0 is the long Weyl group element of W.

The $\bullet$ operation is defined in the Bernstein presentation

$$\mathbf{q}^\bullet = \mathbf{q}, \quad T_w^\bullet = T_{w^{-1}},\ w \in W, \quad \theta_x^\bullet = \theta_x,\ x \in X. \tag{2.3.3}$$

The fact that this definition extends to a star operation is equivalent with the fact that the algebra $\mathcal{H}(\mathbf{q})$ is isomorphic to its opposite algebra.

2.4. The central characters of $\mathcal{H}$ are parametrized by $W-$orbits in the torus $\mathcal{T} := X \otimes_{\mathbb{Z}} \mathbb{C}^*$. In [BM2] (following [Lu1]), a filtration of $\mathcal{H}$ is defined for any finite $W-$invariant set $\mathcal{O} \subset \mathcal{T}$. Let $\mathcal{O}_a$ be the W-orbit of an element $a \in \mathcal{T}$. The θ_x are interpreted as regular functions $R(\mathcal{T})$ on $\mathcal{T}$. The filtration associated to $\mathcal{O}_a$ is defined by the powers of the ideal $\mathcal{I}_a := R(\mathcal{O}_a)\mathcal{H}$ generated by

$$R(\mathcal{O}_a) := \{f \in R(\mathcal{T} \times \mathbb{C}^\times) \ : \ f(\sigma, 1) = 0 \ \text{ for any } \sigma \in \mathcal{O}_a\}. \tag{2.4.1}$$

The graded algebra $\mathbb{H}_a$ is then shown, [Lu1, Proposition 4.4] and [BM2, Proposition 3.2] to be a matrix algebra over an appropriate graded affine Hecke algebra as in Definition 3.1.1.

Let κ be an automorphism (or anti-automorphism) of $\mathcal{H}$. Then κ induces an automorphism of the center $Z(\mathcal{H})$, and therefore an isomorphism

$$\hat{\kappa} : \mathcal{T} \times \mathbb{C}^\times \to \mathcal{T} \times \mathbb{C}^\times.$$

We will only consider morphisms κ that fix $\mathbf{q}$, and thus $\hat{\kappa}$ restricts to an isomorphism of $\mathcal{T}$ as well.

Definition 2.4.1. An automorphism (or anti-automorphism) κ of $\mathcal{H}$ is called admissible, if $\kappa(\mathbf{q}) = \mathbf{q}$, $\kappa(T_w) = T_w$, for all $w \in W$, and $\kappa(\mathcal{I}_a) \subset \mathcal{I}_{\hat{\kappa}(a)}$ for all $a \in \mathcal{T}$. It is clear that the operations $\bullet$ and $\star$ from Definition 2.3.1 are admissible in this sense.

In section 3, we study the analogues of admissible automorphisms and anti-automorphisms for graded affine Hecke algebras. Motivated by the main result of that section, Proposition 3.4.3 and the connection between the affine Hecke algebra $\mathcal{H}$ and the graded Hecke algebras $\mathbb{H}_a$, we make the following conjecture.

Conjecture 2.4.2. *Let κ be an admissible involutive anti-automorphism.*

1. *If $\hat{\kappa}(a) = a$, for all $a \in \mathcal{T}$, then $\kappa(\theta_x) = \theta_x^\bullet$, for all $x \in X$.*
2. *If $\hat{\kappa}(a) = a^{-1}$, for all $a \in \mathcal{T}$, then $\kappa(\theta_x) = \theta_x^\star$, for all $x \in X$.*

2.5. The operation $\star$ appears naturally in relation with smooth representations of reductive p-adic groups. Suppose $\mathbb{F}$ is a p-adic field of characteristic 0 with residue field $\mathbb{F}_q$. Let $\mathcal{G}$ be the group of $\mathbb{F}$-rational points of a connected reductive algebraic group defined over $\mathbb{F}$. Let I be an Iwahori subgroup of $\mathcal{G}$ and let $\mathcal{C}_I(\mathcal{G})$ be the category of smooth admissible $\mathcal{G}$-representations which are generated by their I-fixed vectors.

On the other hand, let $\mathcal{H}(\mathcal{G}, I)$ be the Iwahori-Hecke algebra, i.e., the convolution algebra (with respect to a Haar measure of $\mathcal{G}$) of compactly-supported complex valued functions on G that are I-biinvariant. By a classical result of Borel, the functor $V \mapsto V^I$ induces an equivalence of categories between $\mathcal{C}_I(\mathcal{G})$ and the category of finite dimensional $\mathcal{H}(\mathcal{G}, I)$-modules.

The Iwahori-Hecke algebra $\mathcal{H}(\mathcal{G}, I)$ is a specialization of $\mathcal{H}(\mathbf{q})$ with $\mathbf{q} = \sqrt{q}$ and the appropriate specialization of parameters L , see [Ti]. Under this specialization, the natural star operation

$$f^\star(g) = \overline{f(g^{-1})}, \quad f \in \mathcal{H}(\mathcal{G}, I),$$

on $\mathcal{H}(\mathcal{G}, I)$ corresponds to the operation $\star$ on $\mathcal{H}(\mathbf{q})$.

2.6. **Bernstein's projective modules.** In the rest of this section, we explain how the $\bullet$-form for affine Hecke algebras appears naturally when the Iwahori-Hecke algebras are viewed as endomorphism algebras of the Bernstein projective modules [Be], see also [He].

Let V be a complex vector space,

$$V^h := \left\{ \lambda : V \longrightarrow \mathbb{C} \ : \ \lambda(\alpha_1 v_1 + \alpha_2 v_2) = \overline{\alpha}_1 \lambda(v_1) + \overline{\alpha}_2 \lambda(v_2) \right\}.$$

A sesquilinear form is a bilinear form $\langle \cdot, \cdot \rangle$ which is linear in the first variable, conjugate linear in the second variable. This is the same as a complex linear map $\lambda : V \longrightarrow V^h$. The relation is

$$\langle v, w \rangle_\lambda = \lambda(v)(w).$$

Such a form is called nondegenerate if λ is injective. To any sesquilinear form λ there is associated $\lambda^h : V \subset (V^h)^h \longrightarrow V^h$, $\lambda^h(v)(w) := \overline{\lambda(w)(v)}$. The form is called symmetric, if $\lambda = \lambda^h$. A symmetric form is an inner product if $\lambda(v)(v) \geq 0$, with equality if and only if $v = 0$.

Let G be a reductive p-adic group. If (π, V) is a representation of G, then (π^h, V^h) is the representation defined as

$$\big(\pi^h(g)\lambda\big)(v) := \lambda(\pi(g^{-1})v).$$

2.7. **The projective $\mathcal{P}$.** Let M be a Levi subgroup of G. Denote by M_0 the intersection of the kernels of all the unramified characters of M. Let $\widetilde{\sigma}$ be a relative supercuspidal representation of M, σ_0 a supercuspidal constituent of $\widetilde{\sigma}\mid_{M_0}$.

Define

$$\begin{array}{ll} \big(\sigma,\ V_\sigma = \operatorname{Ind}_{M_0}^{M} \sigma_0\big)_c & \text{induction with compact support,} \\ \big(\Pi,\ \mathcal{P} = \operatorname{Ind}_P^G V_\sigma\big) & \text{normalized induction.} \end{array}$$

A typical element of σ is $\delta_{mM_0,v}$ with $m \in M/M_0$ and $v \in V_{\sigma_0}$. This is the *delta*-function supported on the coset mM_0 taking constant value v.

A typical element of $\mathcal{P}$ is given by $\delta_{UxP,\delta_{mM_0,v}}$ where $U \in G/P$ is a neighborhood of the identity, the function satisfies the appropriate transformation law under P on the right, and the value at x is $\delta_{mM_0,v}$.

If $\psi \in \mathrm{Hom}_G[\mathcal{P},\mathcal{P}]$, then $\psi^h \in \mathrm{Hom}_G[\mathcal{P}^h,\mathcal{P}^h]$. But $\mathcal{P}$ admits a $G-$invariant positive definite hermitian form, so while $\mathcal{P} \neq \mathcal{P}^h$, nevertheless there is an inclusion $\iota : \mathcal{P} \longrightarrow \mathcal{P}^h$. More precisely, if $\mathcal{P} = \mathrm{Ind}_P^G \sigma$, then the hermitian dual $\mathcal{P}^h$ is naturally isomorphic to $\mathrm{Ind}_P^G \sigma^h$. If $\lambda : G \longrightarrow V_\sigma^h$ is such that $\lambda(xp) = \sigma^h(p^{-1})\lambda(g)$, and $f : G \longrightarrow V$ is such that $f(gp) = \sigma(p^{-1})f(x)$, then the pairing is

$$\langle \lambda, f\rangle := \int_{G/P} \lambda(x)(f(x))dx.$$

When σ is unitary (or just has a nondegenerate form so that $\sigma \subset \sigma^h$), we get $\mathcal{P} \subset \mathcal{P}^h$ via

$$g \in \mathcal{P} \mapsto \lambda_g \in \mathcal{P}^h, \quad \lambda_g(f) = \int_{G/P} \langle f(x), g(x)\rangle dx \text{ for } f \in \mathcal{P}.$$

2.8. **Inner Product.** We recall two classical results.

Theorem 2.8.1 (Frobenius reciprocity, [Cas1, Theorem 3.2.4])**.**

$$\mathrm{Hom}_G[V,\mathcal{P}] \cong \mathrm{Hom}_M[V_N, \sigma\delta_P^{-1}].$$

Theorem 2.8.2 (Second adjointness, [Be, Theorem 20])**.**

$$\mathrm{Hom}_G[\mathcal{P},V] \cong \mathrm{Hom}_M[\delta_{\overline{P}}^{-1}\sigma, V_{\overline{N}}] \cong \mathrm{Hom}_{M_0}[\sigma_0, \delta_P^{-1}V_{\overline{N}}].$$

Let $\overline{\mathcal{P}}$ be the module induced from σ from the opposite parabolic $\overline{P} := M\overline{N}$. The (second) adjointness theorem gives

$$\mathrm{Hom}_G[\mathcal{P},\mathcal{P}] = \mathrm{Hom}_M[\delta_{\overline{P}}^{-1}\sigma, \mathcal{P}_{\overline{N}}] = \mathrm{Hom}_{M_0}[\sigma_0, \delta_P^{-1}\mathcal{P}_{\overline{N}}],$$
$$\mathrm{Hom}_G[\overline{\mathcal{P}},\mathcal{P}] = \mathrm{Hom}_M[\delta_{\overline{P}}^{-1}\sigma, \mathcal{P}_N] = \mathrm{Hom}_{M_0}[\sigma_0, \delta_P\mathcal{P}_N].$$

Assume P and $\overline{P}$ are conjugate, and let $w_0 \in W$ be the shortest Weyl group element taking P to $\overline{P}$, stabilizing M and taking N to $\overline{N}$. Assume also that there is an isomorphism $\tau_0 : (\sigma_0, V_{\sigma_0}) \longrightarrow (w_0 \circ \sigma_0, V_{\sigma_0})$. Extend it to $\tau : V_\sigma \longrightarrow V_\sigma$ by $\tau(\delta_{mM_0,v}) = \delta_{w_0(m)M_0,\tau(v)}$. Write $\widetilde{\tau}$ for the isomorphism

$$\widetilde{\tau} : \overline{\mathcal{P}} \longrightarrow \mathcal{P},$$
$$\widetilde{\tau}(f)(x) = \tau(f(xw_0)).$$

Thus given $\Phi, \Psi \in \mathrm{Hom}_G[\mathcal{P},\mathcal{P}]$, then $\overline{\Phi} := \Phi \circ \widetilde{\tau} \in \mathrm{Hom}_G[\overline{\mathcal{P}},\mathcal{P}]$, and they give rise to

$$\overline{\phi} \in \mathrm{Hom}_{M_0}[\sigma_0, \mathcal{P}_{\overline{N}}]$$
$$\psi \in \mathrm{Hom}_{M_0}[\sigma_0, \mathcal{P}_N].$$

According to Casselmann [Cas1, Proposition 4.2.3], there is a nondegenerate pairing $\langle\ ,\ \rangle_{N,\overline{N}}$ between $\mathcal{P}_N$ and $\mathcal{P}_{\overline{N}}$. Given $v_1, v_2 \in V_{\sigma_0}$, we can form

$$\langle v_1, v_2\rangle_{\Phi,\Psi} := \langle\overline{\phi}(v_1), \psi(v_2)\rangle_{N,\overline{N}}.$$

This pairing is invariant and sesquilinear, so there is a constant $m_{\Phi,\Psi}$ such that

$$\langle v_1, v_2\rangle_{\Phi,\Psi} = m_{\Phi,\Psi}\langle v_1, v_2\rangle_{\sigma_0}. \tag{2.8.1}$$

We define a sesquilinear pairing

$$\langle \Phi, \Psi\rangle := m_{\Phi,\Psi}. \tag{2.8.2}$$

2.9. We make the form (2.8.2) precise. Let K_ℓ be an open compact subgroup with an Iwasawa decomposition compatible with P, *i.e.* $K_\ell = K_\ell^- \cdot K_\ell^0 \cdot K_\ell^+$, invariant by w_0.

Let $x_0, y_0 \in V_{\sigma_0}^{K_\ell^0}$, and $x := \delta_{M_0,x_0}, y := \delta_{M_0,y_0}$. Then $\delta_{K_\ell^+\overline{P},x} \in \overline{\mathcal{P}}$ and $\delta_{K_\ell^- P,y} \in \mathcal{P}$. The isomorphism $\widetilde{\tau}$ takes $\delta_{K_\ell^+\overline{P},x}$ to $\delta_{K_\ell^+ w_0 P,\tau(x)}$. So

$$(x_0, y_0)_{\Phi,\Psi} := \langle \overline{\Phi}_{\overline{N}}\big(\delta_{K_\ell^+ w_0 P,\tau(x)}\big), \Psi_N\big(\delta_{K_\ell^- P,y}\big)\rangle_{\overline{N},N} = m_{\Phi,\Psi}\langle x_0, y_0\rangle_{\sigma_0}.$$

Here $\overline{\Phi}_{\overline{N}}$ and Ψ_N are the projection maps onto $\mathcal{P}_{\overline{N}}$ and $\mathcal{P}_N$ respectively.

Let $\Lambda \in A := Z(M)$ be such that it is regular on N and *contracts it.* Let $a(\Lambda)$ and $a(-\Lambda)$ be the K_ℓ double cosets of Λ and its inverse.

By Casselman [Cas1, section 4] and Bernstein [Be, chapter III.3],

$$\mathcal{P}^{a(-\Lambda),K_\ell} \cong \mathcal{P}_{\overline{N}}^{K_\ell^0},$$
$$\mathcal{P}^{a(\Lambda),K_\ell} \cong \mathcal{P}_{N}^{K_\ell^0},$$

because $a(\Lambda)$ contracts K_ℓ^+. We conclude that

$$\begin{aligned} &\delta_{K_\ell P,x} \in \mathcal{P}^{a(-\Lambda),K_\ell}, &&\text{so}\quad \Phi(\delta_{K_\ell P,x}) \in \mathcal{P}^{a(-\Lambda),K_\ell} \cong \mathcal{P}_{\overline{N}}^{K_\ell^0},\\ &\delta_{K_\ell w_0 P,\tau_0 y} \in \mathcal{P}^{a(\Lambda),K_\ell}, &&\text{so}\quad \Psi(\delta_{K_\ell w_0 P,\tau_0 y}) \in \mathcal{P}^{a(\Lambda),K_\ell} \cong \mathcal{P}_{N}^{K_\ell^0}.\end{aligned}$$

Proposition 2.9.1. *With the notation as in (2.8.1), $m_{\Phi,\Psi} = \overline{m_{\Psi,\Phi}}$. In other words, the sesquilinear form (2.8.2) is hermitian.*

Proof. Assume $\tau_0 \neq -Id$, or else use $-\tau_0$. Thus there is x_0 such that $\tau_0 x_0 = x_0$. Let $f_{w_0} := \delta_{K_l w_0 K_l}$. Then $f_{w_0}^* = f_{w_0}$, and

$$\Pi(f_{w_0})\delta_{K_l w_0 P,x} = \delta_{K_\ell P,x},$$
$$\Pi(f_{w_0})\delta_{K_\ell P,x} = \delta_{K_\ell w_0 P,x}.$$

Then

$$\begin{aligned} m_{\Phi,\Psi} < x_0, x_0 > &=< \Phi(\delta_{K_\ell P,x}), \Psi(\delta_{K_\ell w_0 P,x}) >=\\ &=< \Phi(\Pi(f_{w_0})\delta_{K_\ell w_0 P,x}), \Psi(\delta_{K_\ell w_0 P,x}) >=\\ &=< \Phi(\delta_{K_\ell w_0 P,x}), \Psi(\Pi(f_{w_0})\delta_{K_\ell w_0 P,x}) >=\\ &=< \Phi(\delta_{K_\ell w_0 P,x}), \Psi(\delta_{K_\ell P,x}) >=\\ &= \overline{< \Psi(\delta_{K_\ell P,x}), \Phi(\delta_{K_\ell w_0 P,x}) >} =\\ &= \overline{m_{\Psi,\Phi}} < x_0, x_0 > .\end{aligned}$$

∎

2.10. For $a \in A$, let $\Theta_a \in \mathrm{Hom}_G[\mathcal{P}, \mathcal{P}]$ be given by

$$\Theta_a(\delta_{K_\ell gP,x}) = \delta_{K_\ell gP,\theta_a(x)}, \qquad \theta_a(x) := \theta_a(\delta_{mM_0,x_0}) = \delta_{maM_0,x_0}. \tag{2.10.1}$$

Proposition 2.10.1.

$$< \Phi, \Psi \circ \Theta_a >=< \Phi \circ \Theta_a, \Psi > .$$

Proof. There is $f_a \in \mathcal{H}(K_\ell \backslash G/K_\ell)$ (namely $\delta_{K_\ell a K_\ell}$) such that $\Theta_a(\delta_{K_\ell P,x}) = \Pi(f_a)(\delta_{K_\ell P,x})$. Then use the fact that $f_a^* = f_{a^{-1}}$ for $a \in A^+$ *dominant*. □

2.11. **Digression about the intertwining operator.** Let $J : \mathcal{P} \longrightarrow \mathcal{P}$ be given by the formula

$$Jf(x) := \int_N \tau_0 f(xnw_0)\, dn = \int_{\overline{N}} \tau_0 f(xw_0\overline{n})\, d\overline{n}. \tag{2.11.1}$$

This should be considered as a formal expression. When you specialize to a value $\nu \in \widehat{A}$, the split part of the center of M, J will have poles.

Recall the inner product on $\mathcal{P}$,

$$\langle f_1, f_2\rangle := \int_{K_0} \langle f_1(k), f_2(k)\rangle\, dk.$$

Proposition 2.11.1.

$$< Jf_1, f_2 >=< f_1, Jf_2 > .$$

Proof.

$$< f_1, Jf_2 >= \int_{K_0} < f_1(k), \int_{\overline{N}} \tau_0 f_2(kw_0\overline{n})\, d\overline{n} > \ dk. \tag{2.11.2}$$

We can move w_0 and τ_0 to the other side:

$$< f_1, Jf_2 >= \int_{K_0} < \tau_0 f_1(kw_0), \int_{\overline{N}} f_2(k\overline{n})\, d\overline{n} > \ dk. \tag{2.11.3}$$

Write $\overline{n} = \kappa(\overline{n}) \cdot n(\overline{n}) \cdot m(\overline{n})$. So

$$\begin{aligned} &< f_1, Jf_2 >= \int_{K_0} < \tau_0 f_1(kw_0), \int_{\overline{N}} \sigma(m(\overline{n})^{-1}) f_2(k\kappa(\overline{n}))\, d\overline{n} > \ dk = \\ &= \int_{K_0} < \int_{\overline{N}} \sigma(m(\overline{n}))\tau_0 f_1(k\kappa(\overline{n})^{-1}w_0)\, d\overline{n}, f_2(k) > \ dk. \end{aligned} \tag{2.11.4}$$

Since $\kappa(\overline{n}) = \overline{n} \cdot m(\overline{n})^{-1} \cdot n(\overline{n})^{-1}$, we conclude $\kappa(\overline{n})^{-1} = n(\overline{n}) \cdot m(\overline{n}) \cdot \overline{n}^{-1}$. So

$$\begin{aligned} &< f_1, Jf_2 > = \\ &\quad = \int_{K_0} < \int_{\overline{N}} \sigma(m(\overline{n}))\tau_0 f_1(kn(\overline{n})m(\overline{n})\overline{n}^{-1}w_0)\, d\overline{n}, f_2(k) > \ dk = \\ &\quad = \int_{K_0} < \int_{\overline{N}} \tau_0 f_1(kn(\overline{n}))\, d\overline{n}, f_2(k) > \ dk \end{aligned} \tag{2.11.5}$$

because $\sigma(m(\overline{n}))$ is conjugated by w_0 but then flipped back by τ_0, and then cancels $\sigma(m(\overline{n}))$. Finally

$$Jf_1 = \int_{\overline{N}} \tau_0 f_1(kn(\overline{n})w_0)\, d\overline{n}$$

follows from the fact that $\overline{n} \mapsto w_0 n(\overline{n}) w_0^{-1}$ is an isomorphism with trivial Jacobian.

□

2.12. Assume from now on that G is a split p-adic group. Let $P = B = AN$ be a Borel subgroup. Let K_0 be the hyperspecial maximal compact subgroup, and $K_1 \subset \mathcal{I} \subset K_0$ be an Iwahori subgroup. It has an Iwasawa decomposition $\mathcal{I} = \mathcal{I}^- \cdot A_0 \cdot \mathcal{I}^+$. Furthermore, $G = KB = \cup \mathcal{I}wB$ disjoint union where $w \in W$.

We consider the case of the trivial representation of $A_0 := K_0 \cap A$, $\sigma_0 = \mathsf{triv}$, i.e, this is the case of representations with $\mathcal{I}-$fixed vectors. Let $\mathcal{H} = \mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$ be the Iwahori-Hecke algebra of compactly supported smooth $\mathcal{I}$-biinvariant functions with convolution with respect to a Haar measure.

Proposition 2.12.1. *In the Iwahori-spherical case, the algebra* $\mathrm{Hom}[\mathcal{P},\mathcal{P}]$ *is naturally isomorphic to the opposite algebra to* $\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$.

Proof. Recall
$$\mathrm{Hom}_G[\mathcal{P},\mathcal{P}] \cong \mathrm{Hom}_{A_0}[\sigma_0, \mathcal{P}_{\overline{N}}] \cong \mathcal{P}_{\overline{N}}^{A_0} \cong \mathcal{P}^{\mathcal{I}}.$$
The element $\phi_1 = \delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}$ is in $\mathcal{P}^{\mathcal{I}}$, and it generates $\mathcal{P}$. So any $\Phi \in \mathrm{Hom}_G[\mathcal{P},\mathcal{P}]$ is determined by its value on ϕ_1. Furthermore, $\Phi(\phi_1) \in \mathcal{P}^I$.

Conversely, $\phi \in \mathrm{Hom}_{A_0}[\sigma_0, \mathcal{P}_{\overline{N}}] \cong \mathcal{P}_{\overline{N}}^{A_0} \cong \mathcal{P}^{\mathcal{I}}$ gives rise to $\Phi \in \mathrm{Hom}_G[\mathcal{P},\mathcal{P}]$ by the relation
$$\Phi(\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}) = \phi.$$
The map
$$h \in \mathcal{H} \mapsto \Pi(h)\big(\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}\big)$$
is an isomorphism between $\mathcal{H}$ and $\mathcal{P}^{\mathcal{I}}$. Let $h_\psi \in \mathcal{I}$ be the element in $\mathcal{H}$ corresponding to ψ. Then if $\Phi(\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}) = \phi$,
$$\Phi[\psi] = \Phi[\Pi(h_\psi)(\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}})] = \Pi(h_\phi)\Phi[\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}] = \Pi(h_\psi)\phi.$$
Now let $\phi_1, \phi_2 \in \mathcal{P}_{\overline{N}}^{A_0}$. Then
$$(\Phi_1 \circ \Phi_2)(\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}) = \Phi_1[\Pi(h_{\phi_2})(\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}})] = \Pi(h_{\phi_2})\Phi_1[\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}] =$$
$$= \Pi(h_{\phi_2})\Pi(h_{\phi_1})(\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}) = \Pi(h_{\phi_1}) \cdot \Pi(h_{\phi_2})(\delta_{\mathcal{I}^-B,\delta_{A_0,\mathbb{1}}}).$$
□

Remark 2.12.2. The opposite algebra to the Iwahori-Hecke algebra is isomorphic to itself, e.g.,
$$T \circ_{opp} \theta = \theta^{-1} \circ_{opp} T + (q-1)\frac{\theta - \theta^{-1}}{1 - \theta_{-\alpha}}$$
is equivalent to
$$\theta \cdot T = T \cdot \theta^{-1} + (q-1)\frac{\theta - \theta^{-1}}{1 - \theta_{-\alpha}}.$$

2.13. The operators J_α are defined analogously to J for each simple root, integration is along the root subgroup N_α. The operators satisfy the formula analogous to 2.11.1. By specializing to $\nu \in \widehat{A}$ unramified, we can prove the following result. Define
$$F(\Theta) = (q-1)\frac{1}{1 - \Theta^{-1}}, \tag{2.13.1}$$
and write F_α for $F(\Theta_\alpha)$.

Theorem 2.13.1.
$$T_\alpha := J_\alpha - F_\alpha \in \mathrm{Hom}_G[\mathcal{P},\mathcal{P}]. \tag{2.13.2}$$
T_α and Θ_α form a set of generators of $\mathrm{Hom}[\mathcal{P},\mathcal{P}]$ *and satisfy the defining relations in the Bernstein-Lusztig presentation (*[Lu1]*) for the Iwahori-Hecke algebra.*

Sketch of proof. Because the group is split, this reduces to a calculation in $SL(2)$. The operator J has a term which is a rational function in Θ_α with $1-\Theta_{-\alpha}$ in the denominator, and subtracting F_α removes the singularity.

$\square$

Remark 2.13.2. For a classical p-adic group G and any Bernstein projective module $\mathcal{P}$, it is shown in [He] that a generalization of Theorem 2.13.1 holds, namely, $\mathrm{End}_G[\mathcal{P}]$ is naturally isomorphic to an extended affine Hecke algebra with unequal parameters.

Proposition 2.13.3. *There is $f_\alpha \in \mathcal{H}(K_\ell\backslash G/K_\ell)$ and $\tau_\alpha : \sigma \longrightarrow \sigma$ such that*
$$< \Phi(\delta_{K_\ell w_0 P,x}), \Psi(T_\alpha(\delta_{K_\ell P,y})) >=< \Phi(\delta_{K_\ell w_0 P,x}), \Psi\big(\Pi(f_\alpha)(\delta_{K_\ell P,\tau_\alpha(y)})\big) > . \tag{2.13.3}$$

Proof. This follows from the formula of J_α as an integral. We want $T_\alpha(\delta_{K_\ell P,y}) = \Pi(f_\alpha)(\delta_{K_\ell P,y})$.

For $SL(2)$, let K_ℓ be the usual congruence subgroup. Let $a := \begin{bmatrix}\varpi & 0\\ 0 & \varpi^{-1}\end{bmatrix}$. Then $\mathcal{I}B = \mathcal{I}^-B$, and $a^{-\ell}K_\ell B a^\ell = \mathcal{I}^-B$. Thus
$$\Pi(a^{-\ell})(\delta_{K_\ell B,\alpha}) = \delta_{\mathcal{I}a^{-\ell}B,a^\ell\alpha} = \Pi(\delta_{\mathcal{I}a^{-\ell}\mathcal{I}})\delta_{\mathcal{I}B,\alpha}.$$
T_α commutes with $\Pi(\delta_{\mathcal{I}a^{-\ell}\mathcal{I}})$ and $\Pi(a^{-\ell})$, and is computable on $\delta_{\mathcal{I}B,\alpha}$. it can be written as convolution with a $\mathcal{I}-$biinvariant function. The conclusion of the calculation is that $T_\alpha(\delta_{K_\ell B,\alpha})$ can be expressed as convolution with an element $\mathcal{T}_\alpha \in \mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$ and composition with a $\Pi(a^{\pm\ell})$. We can then argue as in Proposition 2.10.1 to conclude that
$$\langle \Phi, \Psi\circ T_\alpha\rangle = \langle \Phi\circ T_\alpha, \Psi\rangle. \tag{2.13.4}$$
$\square$

We summarize the results.

Theorem 2.13.4. *In the case of Iwahori fixed vectors, unramified principal series, $\mathcal{H} := \mathrm{Hom}[\mathcal{P},\mathcal{P}]$ inherits a natural star operation $\bullet$ from the unitary structure of $\mathcal{P}$ satisfying*
$$\langle \Phi, \Psi\circ \mathcal{R}\rangle = \langle \Phi\circ\mathcal{R}^\bullet, \Psi\rangle, \quad \Phi,\Psi,\mathcal{R}\in\mathcal{H}.$$
In particular,
$$T_\alpha^\bullet = T_\alpha, \quad \Theta^\bullet = \Theta.$$

3. Star operations: the graded affine Hecke algebra

3.1. **Graded affine Hecke algebra.** We fix an $\mathbb{R}$-root system $\Phi = (V, R, V^\vee, R^\vee)$. This means that $V, V^\vee$ are finite dimensional $\mathbb{R}$-vector spaces, with a perfect bilinear pairing $(\ ,\ ) : V\times V^\vee \to \mathbb{R}$, where $R\subset V\setminus\{0\}$, $R^\vee\subset V^\vee\setminus\{0\}$ are finite subsets in bijection
$$R\longleftrightarrow R^\vee,\ \alpha\longleftrightarrow\alpha^\vee,\ \text{ satisfying } (\alpha,\alpha^\vee) = 2. \tag{3.1.1}$$

Moreover, the reflections

$$s_\alpha : V \to V,\ s_\alpha(v) = v - (v, \alpha^\vee)\alpha, \quad s_\alpha : V^\vee \to V^\vee,\ s_\alpha(v') = v' - (\alpha, v')\alpha^\vee, \quad \alpha \in R, \tag{3.1.2}$$

leave R and $R^\vee$ invariant, respectively. Let W be the subgroup of $GL(V)$ (respectively $GL(V^\vee)$) generated by $\{s_\alpha :\ \alpha \in R\}$. We assume that the root system Φ is reduced, meaning that $\alpha \in R$ implies $2\alpha \notin R$. We fix a choice of simple roots $\Pi \subset R$, and consequently, positive roots R^+ and positive coroots $R^{\vee,+}$. Often, we will write $\alpha > 0$ or $\alpha < 0$ in place of $\alpha \in R^+$ or $\alpha \in (-R^+)$, respectively. The complexifications of V and $V^\vee$ are denoted by $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$, respectively, and we denote by $\bar{\ }$ the complex conjugations of $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$ induced by V and $V^\vee$, respectively. Extend $(\ ,\ )$ linearly to $V_\mathbb{C} \times V_\mathbb{C}^\vee$. Then

$$\overline{(v, u)} = (\overline{v}, \overline{u}),\ \text{for all } v \in V_\mathbb{C},\ u \in V_\mathbb{C}^\vee. \tag{3.1.3}$$

Let $k : \Pi \to \mathbb{R}$ be a function such that $k_\alpha = k_{\alpha'}$ whenever $\alpha, \alpha' \in \Pi$ are W-conjugate. Let $\mathbb{C}[W]$ denote the group algebra of W and $S(V_\mathbb{C})$ the symmetric algebra over $V_\mathbb{C}$. The group W acts on $S(V_\mathbb{C})$ by extending the action on V. For every $\alpha \in \Pi$, denote the difference operator by

$$\Delta : S(V_\mathbb{C}) \to S(V_\mathbb{C}), \quad \Delta_\alpha(a) = \frac{a - s_\alpha(a)}{\alpha},\ \text{for all } a \in S(V_\mathbb{C}). \tag{3.1.4}$$

Definition 3.1.1. The graded affine Hecke algebra $\mathbb{H} = \mathbb{H}(\Phi, k)$ is the unique associative unital algebra generated by $\mathbb{A} = S(V_\mathbb{C})$ and $\{t_w : w \in W\}$ such that

(i) the assignment $t_w a \mapsto w \otimes a$ gives an isomorphism $\mathbb{H} \cong \mathbb{C}[W] \otimes S(V_\mathbb{C})$ of $(\mathbb{C}[W], S(V_\mathbb{C}))$-bimodules;
(ii) $a t_{s_\alpha} = t_{s_\alpha} s_\alpha(a) + k_\alpha \Delta_\alpha(a)$, for all $\alpha \in \Pi$, $a \in S(V_\mathbb{C})$.

The center of $\mathbb{H}$ is $S(V_\mathbb{C})^W$ ([Lu1]). By Schur's Lemma, the center of $\mathbb{H}$ acts by scalars on each irreducible $\mathbb{H}$-module. The central characters are parameterized by W-orbits in $V_\mathbb{C}^\vee$. If X is an irreducible $\mathbb{H}$-module, denote by $\mathsf{cc}(X) \in W\backslash V_\mathbb{C}^\vee$ its central character. By abuse of notation, we may also denote by $\mathsf{cc}(X)$ a representative in $V_\mathbb{C}^\vee$ of the central character of X.

If (π, X) is a finite dimensional $\mathbb{H}$-module and $\lambda \in V_\mathbb{C}^\vee$, denote

$$X_\lambda = \{x \in X :\ \text{for every } a \in S(V_\mathbb{C}),\ (\pi(a) - (a, \lambda))^n x = 0,\ \text{for some } n \in \mathbb{N}\}. \tag{3.1.5}$$

If $X_\lambda \neq 0$, call λ an $\mathbb{A}$-weight of X. Let $\Omega(X) \subset V_\mathbb{C}^\vee$ denote the set of $\mathbb{A}$-weights of X. If X has a central character, it is easy to see that $\Omega(X) \subset W \cdot \mathsf{cc}(X)$.

Definition 3.1.2 (Casselman's criterion)**.** Set

$$V^+ = \{\omega \in V : (\omega, \alpha^\vee) > 0,\ \text{for all } \alpha \in \Pi\}.$$

An irreducible $\mathbb{H}$-module X is called tempered if

$$(\omega, \Re\lambda) \leq 0,\ \text{for all } \lambda \in \Omega(X) \text{ and all } \omega \in V^+.$$

A tempered module is called a discrete series module if all the inequalities are strict.

When the root system Φ is semisimple, $\mathbb{H}$ has a particular discrete series module, the Steinberg module St. This is a one-dimensional module, on which W acts via the sgn representation, and the only $\mathbb{A}$-weight is $-\sum_{\alpha \in \Pi} k_\alpha \omega_\alpha^\vee$, where $\omega_\alpha^\vee$ is the fundamental coweight corresponding to α.

3.2. **An automorphism of $\mathbb{H}$.** Let w_0 denote the long Weyl group element. Define an assignment

$$\delta(t_w)=t_{w_0ww_0},\ w\in W,\quad \delta(\omega)=-w_0(\omega),\ \omega\in V_{\mathbb{C}}. \tag{3.2.1}$$

Lemma 3.2.1. *Suppose $k_{\delta(\alpha)}=k_\alpha$, for all $\alpha\in\Pi$. The assignment δ from (3.2.1) extends to an involutive automorphism of $\mathbb{H}$. When w_0 is central in W, $\delta=\mathrm{Id}$.*

Proof. It is clear that δ is an automorphism of $\mathbb{C}[W]$ and it also extends to an automorphism on $S(V_{\mathbb{C}})$, so it remains to check the commutation relation in Definition 3.1.1:

$$\omega t_{s_\alpha}-t_{s_\alpha}s_\alpha(\omega)=k_\alpha(\omega,\alpha^\vee),\ \alpha\in\Pi,\ \omega\in V_{\mathbb{C}}. \tag{3.2.2}$$

Then

$$\begin{aligned}\delta(\omega)\delta(t_{s_\alpha})=&\delta(\omega)t_{s_{\delta(\alpha)}}=t_{s_{\delta(\alpha)}}s_{\delta(\alpha)}(\delta(\omega))+k_{\delta(\alpha)}(\delta(\omega),\delta(\alpha)^\vee)=\\ =&t_{s_{\delta(\alpha)}}s_{\delta(\alpha)}(\delta(\omega))+k_\alpha(\omega,\alpha^\vee).\end{aligned}$$

Notice that we have used the fact that $\delta(\alpha)\in\Pi$ if $\alpha\in\Pi$. It is easy to see that $\delta(s_\alpha(\omega))=s_{\delta(\alpha)}(\delta(\omega))$.

Since $w_0^2=1$, $\delta^2=\mathrm{Id}$. □

Thus, one may define an extended graded Hecke algebra $\mathbb{H}'=\mathbb{H}\rtimes\langle\delta\rangle$.

3.3. **Star operations.**

Definition 3.3.1. Let $\kappa:\mathbb{H}\to\mathbb{H}$ be a conjugate linear involutive algebra anti-automorphism. An $\mathbb{H}$-module (π,X) is said to be κ-hermitian if X has a hermitian form $(\ ,\ )$ which is κ-invariant, i.e.,

$$(\pi(h)x,y)=(x,\pi(\kappa(h))y),\quad x,y\in X,\ h\in\mathbb{H}.$$

A hermitian module X is κ-unitary if the κ-hermitian form is positive definite.

Definition 3.3.2. Define

$$t_w^\star=t_{w^{-1}},\ w\in W,\quad \omega^\star=-t_{w_0}\cdot\overline{\delta(\omega)}\cdot t_{w_0}=(\mathrm{Ad}\,t_{w_0}\circ\delta)(\overline{\omega}),\ \omega\in V_{\mathbb{C}}, \tag{3.3.1}$$

and

$$t_w^\bullet=t_{w^{-1}},\ w\in W,\quad \omega^\bullet=\overline{\omega},\ \omega\in V_{\mathbb{C}}. \tag{3.3.2}$$

Lemma 3.3.3. *The operations $\star$ and $\bullet$ defined in (3.3.1) and (3.3.2), respectively, extend to conjugate linear algebra anti-involutions of $\mathbb{H}$.*

Proof. Straightforward by Lemma 3.2.1. □

Remark 3.3.4. The two star operations just defined are related as follows

$$\star=(\mathrm{Ad}\,t_{w_0}\circ\delta)(h)\circ\bullet. \tag{3.3.3}$$

In particular, when w_0 is central in W, they are inner conjugate to each other.

Lemma 3.3.5. *For every $w\in W$, $\omega\in V_{\mathbb{C}}$,*

$$t_w\cdot\omega\cdot t_{w^{-1}}=w(\omega)+\sum_{\beta>0,w(\beta)<0}k_\beta(\omega,\beta^\vee)t_{s_{w(\beta)}}. \tag{3.3.4}$$

In particular,

$$\omega^\star=-\overline{\omega}+\sum_{\beta>0}k_\beta(\overline{\omega},\beta^\vee)t_{s_\beta}. \tag{3.3.5}$$

Proof. This is [BM2, Theorem 5.6]. □

3.4. **Classification of involutions.** We define a filtration of $\mathbb{H}$ given by the degree in $S(V_{\mathbb{C}})$. Set $\deg t_w a = \deg_{S(V_{\mathbb{C}})} a$ for every $w \in W$, and homogeneous element $a \in S(V_{\mathbb{C}})$ and $F_i\mathbb{H} = \operatorname{span}\{h \in \mathbb{H} : \deg h \leq i\}$. In particular, $F_0\mathbb{H} = \mathbb{C}[W]$. Set $F_{-1}\mathbb{H} = 0$. It is immediate from Definition 3.1.1 that the associated graded algebra $\overline{\mathbb{H}} = \oplus_{i\geq 0}\overline{\mathbb{H}}^i$, where $\overline{\mathbb{H}}^i = F_i\mathbb{H}/F_{i-1}\mathbb{H}$, is naturally isomorphic to the graded Hecke algebra for the parameter function $k_\alpha \equiv 0$.

Definition 3.4.1. An automorphism (respectively, anti-automorphim) κ of $\mathbb{H}$ is called *filtered* if $\kappa(F_i\mathbb{H}) \subset F_i\mathbb{H}$, for all $i \geq 0$. Notice that by Definition 3.1.1, this is equivalent with the requirement that $\kappa(F_i\mathbb{H}) \subset F_i\mathbb{H}$ for $i = 0, 1$. If, in addition, $\kappa(t_w) = t_w$ (resp., $\kappa(t_w) = t_{w^{-1}}$), we say that κ is *admissible.*

If κ is a filtered automorphism, then κ induces an automorphism of the associated graded algebra $\overline{\mathbb{H}}$ which preserves that grading, i.e., $\kappa(\overline{\mathbb{H}}^i) \subset \overline{\mathbb{H}}^i$.

Lemma 3.4.2. *Assume the root system Φ is simple. Let κ be an admissible involutive automorphism (or anti-automorphism) of $\overline{\mathbb{H}}$ which preserves the grading $\kappa(\overline{\mathbb{H}}^i) \subset \overline{\mathbb{H}}^i$. Then $\kappa(\omega) = c_0\omega$, for all $\omega \in V_{\mathbb{C}}$, where c_0 is a constant equal to 1 or -1.*

Proof. We prove the statement in the case when κ is an automorphism. Since $\mathbb{H}$ is isomorphic to the opposite algebra $\mathbb{H}^{\mathsf{opp}}$ via the map $\tau : t_w \mapsto t_{w^{-1}}$, $\omega \mapsto \omega$, the classification of anti-automorphisms follows by composition with τ.

By the assumptions on κ,

$$\kappa(\omega) = \sum_{y\in W} f_y(\omega)t_y,\ \omega \in V_{\mathbb{C}}, \tag{3.4.1}$$

where $f_y : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is a linear function, for every $y \in W$. Let α be a simple root. The commutation relation in $\overline{\mathbb{H}}$ is $t_{s_\alpha}\omega = s_\alpha(\omega)t_{s_\alpha}$. Applying κ to this relation, it follows, by a simple calculation, that

$$s_\alpha(f_{s_\alpha x}(\omega)) = f_{xs_\alpha}(s_\alpha(\omega)),\ \text{for all } x \in W.$$

In particular, setting $x = s_\alpha$, we see that

$$s_\alpha(f_1(\omega)) = f_1(s_\alpha(\omega)). \tag{3.4.2}$$

Since the root system was assumed simple, this means that f_1 is a scalar function $f_1(\omega) = c_0\omega$, for some $c_0 \in \mathbb{C}$.

Now, we use that κ is an involution, $\kappa^2(\omega) = \omega$, which implies $\sum_{x,y\in W}(f_x \circ f_y)(\omega)t_{xy} = \omega$. Thus

$$\sum_{x\in W} f_x \circ f_{x^{-1}} = \mathrm{Id},\ \text{and } f_x \circ f_y = 0,\ \text{if } x \neq y^{-1}. \tag{3.4.3}$$

Specializing $y = 1$ in the second relation, we see that $f_x = 0$ if $x \neq 1$. Then the first relation implies $c_0^2 = 1$, and this is the claim of the lemma. □

Proposition 3.4.3. *Assume the root system Φ is simple. If κ is an admissible involutive automorphism or anti-automorphism (in the sense of Definition 3.4.1), then*

$$\kappa(\omega) = \omega,\ \textit{for all } \omega \in V,$$

or

$$\kappa(\omega) = t_{w_0} \cdot \delta(\omega) \cdot t_{w_0},\ \textit{for all } \omega \in V.$$

In particular, the only admissible conjugate linear involutive anti-automorphisms of $\mathbb{H}$ are $\star$ and $\bullet$ from Lemma 3.3.3.

Proof. As before, it is sufficient to only treat the case when κ is an automorphism. The hypotheses imply that $\kappa^2 = \mathrm{Id}$, and in addition, by Lemma 3.4.2, κ must be of the form:

$$\kappa(t_w) = t_w, \quad w \in W; \qquad\qquad \kappa(\omega) = c_0\omega + \sum_{y\in W} g_y(\omega)t_y, \quad \omega \in V_{\mathbb{C}},$$

where $g_y : V_{\mathbb{C}} \to \mathbb{C}$, $y \in W$, are linear, and $c_0 = \pm 1$. Since κ has to preserve the commutation relation

$$t_{s_\alpha}\omega - s_\alpha(\omega)t_{s_\alpha} = k_\alpha(\omega, \alpha^\vee),\ \alpha \in \Pi, \omega \in V_{\mathbb{C}},$$

we find that

$$c_0 t_{s_\alpha}\omega - c_0 s_\alpha(\omega)t_{s_\alpha} + \sum_{y\in W} g_y(\omega)t_{s_\alpha y} - \sum_{x\in W} g_x(s_\alpha(\omega))t_{xs_\alpha} = k_\alpha(\omega, \alpha^\vee),$$

or equivalently,

$$\sum_{y\in W} g_y(\omega)t_{s_\alpha y} - \sum_{x\in W} g_x(s_\alpha(\omega))t_{xs_\alpha} = k_\alpha(1-c_0)(\omega, \alpha^\vee). \tag{3.4.4}$$

This implies that

$$g_{s_\alpha y s_\alpha}(\omega) = g_y(s_\alpha(\omega)),\ \text{for all } \alpha \in \Pi, y \in W, y \neq s_\alpha,\ \text{and } \omega \in V_{\mathbb{C}}, \tag{3.4.5}$$

and

$$g_{s_\alpha}(\omega) - g_{s_\alpha}(s_\alpha(\omega)) = k_\alpha(1-c_0)(\omega, \alpha^\vee),$$

from which one easily concludes that

$$g_{s_\alpha}(\alpha) = k_\alpha(1-c_0),\ \alpha \in \Pi. \tag{3.4.6}$$

There are two cases:

(1) $c_0 = 1$,
(2) $c_0 = -1$.

In case (1), $g_y = 0$ for all y, so $\kappa(\omega) = \omega$, $\omega \in V$.

When $w_0 = -\mathrm{Id}$, we note that since $\kappa(t_{w_0}) = t_{w_0}$, $\kappa' := \kappa \circ \mathrm{Ad}\, t_{w_0} = \mathrm{Ad}\, t_{w_0} \circ \kappa$ is as in case (1).

When $w_0 \neq -\mathrm{Id}$, recall δ the automorphism defined in (3.2.1). If we knew that $\kappa \circ \delta = \delta \circ \kappa$, the same proof would apply, since $\delta \circ \mathrm{Ad}\, t_{w_0} \circ \kappa$ is of the same type as κ, but c_0 changes to $-c_0$. Since this is not clear to us, we prove directly that in case (2), $\kappa(\omega) = t_{w_0} \cdot \delta(\omega) \cdot t_{w_0}$.

We first show that $g_y = 0$ unless $y = s_\beta$ for some positive root β. If $y = 1$, relation (3.4.5) shows that $g_y = 0$, so assume $y \neq 1$. The automorphism κ must also satisfy $\kappa(\omega_1)\kappa(\omega_2) = \kappa(\omega_2)\kappa(\omega_1)$ for all $\omega_1, \omega_2 \in V_{\mathbb{C}}$. This implies that

$$g_y(\omega_2)(\omega_1 - y^{-1}(\omega_1)) = g_y(\omega_1)(\omega_2 - y^{-1}(\omega_2)),\ \text{for all } y \in W,\ \omega_1, \omega_2 \in V_{\mathbb{C}}. \tag{3.4.7}$$

If λ_1, λ_2 are eigenvalues of y^{-1}, then for $\omega_1 \in V_{\lambda_1}$ $\omega_2 \in V_{\lambda_2}$,

$$g_y(\omega_1)(1-\lambda_2)\omega_2 = g_y(\omega_2)(1-\lambda_1)\omega_1. \tag{3.4.8}$$

Set $\lambda_1 = 1$. Then

$$g_y(\omega_1)(1-\lambda_2)\omega_2 = 0 \text{ for any } \omega_2 \in V_{\lambda_2}.$$

Because $y^{-1} \neq 1$, it has an eigenvalue $\lambda_2 \neq 1$, so g_y is 0 on the $1-$eigenspace of y^{-1}. Similarly, relation (3.4.8) implies that if $\lambda \neq 1$, any $\omega_1, \omega_2 \in V_\lambda$ must be multiples of each other. So $\dim V_\lambda \leq 1$ for any $\lambda \neq 1$.

Because y is an automorphism of the real space V, if λ is an eigenvalue, so is $\overline{\lambda}$. From relation (3.4.8), we see that unless $\lambda = \overline{\lambda}$, $g_y = 0$ on these eigenspaces. The only remaining case, when $g_y \neq 0$, is when y^{-1} has eigenvalues ± 1, and the $-1-$eigenspace has dimension 1. It follows that $g_y = 0$ unless $y = s_\beta$ for a root β.

We specialize $y = s_\beta$, for $\beta \in R^+$. Then

$$g_{s_\beta}(\omega_2)(\omega_1, \beta^\vee)\beta = g_{s_\beta}(\omega_1)(\omega_2, \beta^\vee)\beta,\ \omega_1, \omega_2 \in V_{\mathbb{C}},$$

and therefore $g_{s_\beta}(\omega) = c_\beta(\omega, \beta^\vee)$, for some $c_\beta \in \mathbb{C}$. When $\beta = \alpha \in \Pi$, (3.4.6) with $c_0 = -1$, implies that $c_\alpha = k_\alpha$. If β is not a simple root, we can use (3.4.5) inductively to check that $c_\beta = k_\beta$.

□

Remark 3.4.4. There may be many more (up to inner conjugation) filtered automorphisms κ that preserve, but are not the identity on W. Every filtered automorphism κ induces an automorphism of $\mathbb{C}[W]$, so a first question would be to classify the group of outer automorphisms of $\mathbb{C}[W]$, a subgroup of which is $\mathsf{Out}(W)$, and this can be nontrivial (e.g., when $W = S_6$, $\mathsf{Out}(S_6) = \mathbb{Z}/2\mathbb{Z}$). But if we require that κ preserves the root reflections, then κ is obtained from one of the two automorphisms in Proposition 3.4.3 by composition with an automorphism of $\mathbb{H}$ coming from the root system.

4. Relation between signatures

In this section, we discuss the relation between the signature characters for $\star$ and $\bullet$ of simple hermitian $\mathbb{H}$-modules.

4.1. Let $\mathbb{H}' = \mathbb{H} \rtimes \langle\delta\rangle$ be the extended graded Hecke algebra and (π, X) a module for $\mathbb{H}'$. Then, X has a $\bullet$-invariant form if and only if it has a $\star$-invariant form, see [BC4, Lemma 3.1.1] and the relation between the forms is

$$\langle x, y\rangle_\star = \langle x, \pi(t_{w_0}\delta)y\rangle_\bullet. \tag{4.1.1}$$

For example, this applies to the case when X is a simple $\mathbb{H}$-module with real central character. In that case, let (μ, U_μ) be a lowest W-type of X, and extend X to a $\mathbb{H}'$-module, as we may, by normalizing the action of δ so that $\pi(t_{w_0}\delta)$ acts on μ by the identity. (Since $t_{w_0}\delta$ is central in $W' = W \rtimes \langle\delta\rangle$, a priori, it acts on μ by $\pm\,\mathrm{Id}$ depending how μ is extended to a W'-type.)

Define the elements

$$\widetilde{\omega} = \frac{1}{2}(\omega - \omega^\star) = \omega - \frac{1}{2}\sum_{\beta>0}\Delta_\beta(\omega)t_{s_\beta}, \quad \omega \in V. \tag{4.1.2}$$

These elements satisfy:

(1) $\widetilde{\omega}^\star = -\widetilde{\omega}$, $\widetilde{\omega}^\bullet = \widetilde{\omega}$;
(2) $t_w\widetilde{\omega}t_{w^{-1}} = \widetilde{w(\omega)}$;
(3) $[\widetilde{\omega}_1, \widetilde{\omega}_2] = -[T_{\omega_1}, T_{\omega_2}] \in \mathbb{C}[W]$, where $T_\omega = \frac{1}{2}\sum_{\beta>0}\Delta_\beta(\omega)t_{s_\beta}$.

Define the following increasing filtration on X:

$$\mathcal{F}^k X = \operatorname{span}\{\pi(\widetilde{\omega}_1 \cdot \dots \cdot \widetilde{\omega}_j)u : u \in U_\mu,\ \omega_i \in V,\ j \le k\}, \quad k \ge 0. \tag{4.1.3}$$

This filtration is W'-invariant by (2) above, and obviously finite if X is finite dimensional. It depends on the chosen lowest W-type μ. One can define this filtration by starting with any W-invariant subspace of X in degree 0, for example, by replacing V_μ with the sum of lowest W-types.

Let $\oplus\overline{\mathcal{F}}^k X$ be the associated graded object, each $\overline{\mathcal{F}}^k X$ is a W-module. Since

$$\pi(t_{w_0}\delta)\pi(\widetilde{\omega}_1 \cdot \dots \cdot \widetilde{\omega}_j)u = (-1)^k \pi(\widetilde{\omega}_1 \cdot \dots \cdot \widetilde{\omega}_j)\pi(t_{w_0}\delta)u = (-1)^k \pi(\widetilde{\omega}_1 \cdot \dots \cdot \widetilde{\omega}_j)u,$$

$t_{w_0}\delta$ acts by $(-1)^k$ on $\overline{\mathcal{F}}^k X$. This means that $\langle \overline{\mathcal{F}}^k X, \overline{\mathcal{F}}^\ell X\rangle_\bullet = 0$ if $k \not\equiv \ell \pmod 2$ and moreover, we have the following relation.

Lemma 4.1.1. *If $x \in X$, let $k(x)$ be the integer such that $0 \ne \overline{x} \in \overline{\mathcal{F}}^{k(x)} X$. Then*

$$\langle x, y\rangle_\star = \begin{cases} \langle x, y\rangle_\bullet = 0, & \text{if } k(x) \not\equiv k(y) \ (\text{mod } 2), \\ (-1)^{k(x)}\langle x, y\rangle_\bullet, & \text{if } k(x) \equiv k(y) \ (\text{mod } 2). \end{cases}$$

4.2. Denote

$$\overline{X}_0 = \sum_{k \text{ even}} \overline{\mathcal{F}}^k X \text{ and } \overline{X}_1 = \sum_{k \text{ odd}} \overline{\mathcal{F}}^k X. \tag{4.2.1}$$

Notice that $\overline{X}_0$ and $\overline{X}_1$ are the $+1$ and -1 eigenspaces of $t_{w_0}\delta$ in X, so they do not depend on the chosen filtration. The previous lemma implies that a necessary condition for the module X to be $\star$-unitary is that the $\bullet$-form be positive definite on $\overline{X}_0$ and negative definite on $\overline{X}_1$.

Let $\mathbb{H}_{\mathsf{ev}}$ be the subalgebra of $\mathbb{H}$ generated by W and $\{\widetilde{\omega}_1\widetilde{\omega}_2 : \omega_1, \omega_2 \in V\}$ and $\mathbb{H}'_{\mathsf{ev}} = \mathbb{H}_{\mathsf{ev}} \rtimes \langle\delta\rangle$. Then $\overline{X}_0$ and $\overline{X}_1$ are both $\mathbb{H}'_{\mathsf{ev}}$-modules, and with the inherited $\bullet$-form, they are $\bullet$-unitarizable $\mathbb{H}'_{\mathsf{ev}}$-modules. Notice also that if X is a simple $\mathbb{H}'$-module, then $\overline{X}_0$ and $\overline{X}_1$ are simple $\mathbb{H}'_{\mathsf{ev}}$-modules.

In conclusion:

Lemma 4.2.1. *A necessary condition for the simple $\mathbb{H}'$-module X to be $\star$-unitary is that $\overline{X}_0$ and $\overline{X}_1$ be $\bullet$-unitarizable simple $\mathbb{H}'_{\mathsf{ev}}$-modules (or zero).*

Example 4.2.2. Let $\mathbb{H}$ be the graded algebra of type A_1 with generators t and ω: $t\omega + \omega t = 2$. Then $\widetilde{\omega} = \omega - t$ and $\mathbb{H}_{\mathsf{ev}}$ is generated by t and ω^2. Since $t\omega^2 = \omega^2 t$, the simple $\mathbb{H}_{\mathsf{ev}}$-modules are one-dimensional of the form $X(\mathsf{triv}, \lambda)$ or $X(\mathsf{sgn}, \lambda)$, where the restriction to W is triv or sgn, respectively, and ω^2 acts by λ. Suppose λ is real and let x_λ be a generator of such a module X. Then we can define a positive definite $\bullet$-invariant form on X by setting $\langle x_\lambda, x_\lambda\rangle_\bullet = 1$.

Example 4.2.3. Suppose (π, X) is a simple tempered $\mathbb{H}$-module with real central character. Let $\mathfrak{g}$ be the complex Lie algebra attached to the root system and $G = \operatorname{Ad}\mathfrak{g}$. By [KL, Lu2], there exists a nilpotent element $e \in \mathfrak{g}$ and $\psi \in \widehat{A_G(e)}$ of Springer type such that

$$X|_W = H^*(\mathcal{B}_e)^\psi = \sum_{i=0}^{d_e} H^{2i}(\mathcal{B}_e)^\psi.$$

To emphasize the connection write $X(e, \psi)$ for X. Then $X(e, \psi)$ has a unique lowest W-type, namely $\mu(e, \psi) = H^{2d_e}(\mathcal{B}_e)^\psi$, and we define the filtration accordingly. One

can define an action of δ on $H^*(\mathcal{B}_e)^\psi$ (see [CH, section 4] or [BeMi]), which makes $H^*(\mathcal{B}_e)^\psi$ into a W'-module and

$$\operatorname{tr}(ww_0\delta, H^{2i}(\mathcal{B}_e)^\psi) = (-1)^i \mathsf{sgn}(w_0)\operatorname{tr}(w, H^{2i}(\mathcal{B}_e)^\psi). \tag{4.2.2}$$

Moreover, this action is compatible with the $\mathbb{H}$-action on $X(e,\psi)$ [CH, section 6.4], making $X(e,\psi)$ into an $\mathbb{H}'$-module. Thus, once we normalized the action so that δ acts by Id on $\mu(e,\psi)$, we have

$$\overline{X}_0 = \sum_{0\le i\le d_e,\ i\equiv d_e \pmod 2} H^{2i}(\mathcal{B}_e)^\psi \text{ and } \overline{X}_1 = \sum_{0\le i\le d_e,\ i\not\equiv d_e \pmod 2} H^{2i}(\mathcal{B}_e)^\psi. \tag{4.2.3}$$

Example 4.2.4. If $\overline{X}_0 = X$ then X must be a one-W-type in the sense of [BM4]. The one-W-type modules are the only simple $\mathbb{H}$-modules with real central character which are unitary with respect to both $\bullet$ and $\star$ operations. This follows from an argument which is essentially in [BM4, Proposition 2.3], see [CM, Proposition 3.1.1].

5. Invariant forms on spherical principal series

5.1. **Spherical principal series.** In this section, we define $\star$- and $\bullet$-invariant hermitian forms on spherical principal series $\mathbb{H}$-modules (when such forms exist).

Every element $h\in\mathbb{H}$ can be written uniquely as $h=\sum_{w\in W} t_w a_w$, $a_w\in S(V_\mathbb{C})$. Define the $\mathbb{C}$-linear map

$$\epsilon_A : \mathbb{H}\to S(V_\mathbb{C}),\quad \epsilon_A(h) = a_1.$$

If $\nu\in V_\mathbb{C}^\vee$, let $\mathbb{C}_\nu$ denote the character of $S(V_\mathbb{C})$ given by evaluation at ν. For $a\in\mathbb{H}$, denote by $a(\nu)$ the evaluation of a at ν. The spherical principal series with parameter ν is

$$X(\nu) = \mathbb{H}\otimes_{S(V_\mathbb{C})}\mathbb{C}_\nu.$$

If κ is any conjugate linear anti-involution of $\mathbb{H}$, and L, R are arbitrary elements of $\mathbb{H}$, and $\nu'\in V_\mathbb{C}^\vee$, the assignment

$$\langle h_1,h_2\rangle_{L,R} = \epsilon_A(L\kappa(h_2)h_1R)(\nu'),\quad h_1,h_2\in\mathbb{H}, \tag{5.1.1}$$

defines a κ-invariant (not necessarily hermitian) pairing on $\mathbb{H}$ viewed as an $\mathbb{H}$-module under left multiplication. We will omit the subscript L, R from the notation. For such a form to descend to a κ-invariant hermitian form on $X(\nu)$, it must satisfy:

- (H1) $\langle h_1 a, h_2\rangle = a(\nu)\langle h_1,h_2\rangle$, for all $a\in S(V_\mathbb{C})$;
- (H2) $\langle h_1, h_2 a\rangle = \overline{a(\nu)}\langle h_1,h_2\rangle$, for all $a\in S(V_\mathbb{C})$;
- (H3) $\langle h_1,h_2\rangle = \overline{\langle h_2,h_1\rangle}$.

Of course, (H1) and (H3) imply (H2), but in practice it will be convenient for us to check (1) and (2) first, which will then reduce the verification of (3) on the basis $\{t_w\in W\}$ of $X(\nu)$.

For every $s_\alpha\in W$, $\alpha\in\Pi$, define

$$R_{s_\alpha} = (t_{s_\alpha}\alpha - k_\alpha)(\alpha-k_\alpha)^{-1}. \tag{5.1.2}$$

As it is well known, the elements R_{s_α} satisfy the braid relations, therefore one can define R_x, $x\in W$, as a product, using a reduced expression of x. The main property of R_x is that

$$a\cdot R_x = R_x\cdot x^{-1}(a),\ \text{for all } x\in W,\ a\in S(V_\mathbb{C}). \tag{5.1.3}$$

We show (H1)-(H3) for $\kappa = \bullet$ and the pairing

$$\langle h_1, h_2 \rangle_\bullet := \epsilon_A(t_{w_0} h_2^\bullet h_1 R_{w_0})(w_0\nu). \tag{5.1.4}$$

Let

$$\mathcal{R}_\alpha := (t_\alpha \alpha - k_\alpha)(k_\alpha + \alpha)^{-1},$$

and for $x = s_{\alpha_1} \dots s_{\alpha_k}$, define $\mathcal{R}_x = \prod \mathcal{R}_{\alpha_i}$. The $\mathcal{R}_x$ have the same commutation properties as the R_x, and

$$\mathcal{R}_x^\bullet = (-1)^{\ell(x)} \mathcal{R}_{x^{-1}} \prod_{x^{-1}\alpha<0} \frac{k_\alpha + \alpha}{k_\alpha - \alpha}. \tag{5.1.5}$$

Let

$$V^\vee_{\mathsf{reg}} := \{\nu \in V^\vee_{\mathbb{C}} \ : \ (\alpha, \nu) \neq 0 \text{ for any } \alpha \in R^+\}.$$

For $\nu \in V^\vee_{\mathsf{reg}}$, a basis of $X(\nu)$ is given by

$$\{\mathcal{R}_x \otimes \mathbb{1}_\nu\}_{x \in W}. \tag{5.1.6}$$

Notice that $\mathcal{R}_x$ is not in $\mathbb{H}$, but in $\hat{\mathbb{H}}$. However it makes sense to express $\mathcal{R}_x = \sum t_y a^x_y$ with $a^x_y \in \mathcal{O}(V_\mathbb{C})$, and then evaluate at ν. The fact that $\nu \in V^\vee_{\mathsf{reg}}$ allows one to solve for the $t_x \otimes \mathbb{1}_\nu$ in terms of the $\mathcal{R}_x \otimes \mathbb{1}_\nu$; so indeed (5.1.6) is a basis. (Note that we have assumed that $k_\alpha > 0$.)

Lemma 5.1.1. *The vector $\mathcal{R}_x \otimes \mathbb{1}_\nu$ is an $\mathbb{A}$-weight vector of $X(\nu)$ with weight $x\nu$.*

Proof. Since $a \cdot \mathcal{R}_x = \mathcal{R}_x \cdot x^{-1}(a)$, $a \in S(V_\mathbb{C})$, it follows that in $X(\nu)$, $a \cdot (\mathcal{R}_x \otimes \mathbb{1}_\nu) = a(x\nu)(\mathcal{R}_x \otimes \mathbb{1}_\nu)$. □

We show that (H1)-(H3) hold for (5.1.6) and $\nu \in V^\vee_{\mathsf{reg}}$. Since the relations (and the change of basis matrices to the t_x) are rational in ν, and $V^\vee_{\mathsf{reg}}$ contains an open set in $V^\vee_\mathbb{C}$, they will hold in general.

The first identity holds by (5.1.3):

$$\langle h_1 a, h_2 \rangle_\bullet = \langle h_1, h_2 \rangle_\bullet a(\nu).$$

For the second identity,

$$\langle \mathcal{R}_x, \mathcal{R}_y a \rangle_\bullet = \langle \mathcal{R}_x, \mathcal{R}_y \rangle_\bullet (w_0 x^{-1} y)(a^\bullet)(w_0\nu) = \langle \mathcal{R}_x, \mathcal{R}_y \rangle_\bullet (x^{-1}y)(a^\bullet)(\nu).$$

Suppose $x = y$. Then this formula implies (H2) (with $h_1 = h_2 = \mathcal{R}_x$) if and only if $a^\bullet(\nu) = \overline{a(\nu)}$ which is equivalent to $\nu = \overline{\nu}$, i.e., $\nu \in V^\vee$.

Suppose $x \neq y$. We show that each of the two sides of (H2) are zero because $\epsilon_A(t_{w_0} \mathcal{R}_z R_{w_0}) = 0$ unless $z = 1$:

$$\epsilon_A\left(t_{w_0} (\mathcal{R}_y a)^\bullet \mathcal{R}_x R_{w_0}\right) = \epsilon_A\left(t_{w_0} a (-1)^{\ell(y)} \mathcal{R}_{y^{-1}} \prod_{y^{-1}\alpha<0} \frac{k_\alpha + \alpha}{k_\alpha - \alpha} \mathcal{R}_x R_{w_0}\right) =$$

$$= \epsilon_A\left(t_{w_0} \mathcal{R}_{y^{-1}x} R_{w_0}\right) \cdot (-1)^{\ell(y)} (w_0 x^{-1} y)(a) \prod_{y^{-1}\alpha<0} \frac{k_\alpha + w_0 x^{-1}\alpha}{k_\alpha - w_0 x^{-1}\alpha} = 0, \text{ and}$$

$$\epsilon_A\left(t_{w_0} \mathcal{R}_y^\bullet \mathcal{R}_x R_{w_0}\right) a = \epsilon_A\left(t_{w_0} \mathcal{R}_{y^{-1}x} R_{w_0}\right) \cdot (-1)^{\ell(x)} \prod_{y^{-1}\alpha<0} \frac{k_\alpha + w_0 x^{-1}\alpha}{k_\alpha - w_0 x^{-1}\alpha} a = 0.$$

So (H2) is verified.

We also record the formula

$$\begin{aligned}\langle \mathcal{R}_x \otimes \mathbb{1}_\nu, \mathcal{R}_x \otimes \mathbb{1}_\nu\rangle_\bullet &= (-1)^{\ell(x)} \left(\prod_{\alpha>0} \frac{\alpha}{k_\alpha - \alpha} \cdot \prod_{x^{-1}\alpha<0} \frac{k_\alpha - \delta(x^{-1}\alpha)}{k_\alpha + \delta(x^{-1}\alpha)}\right)(w_0\nu) \\ &= (-1)^{|R|} \prod_{\alpha>0} \frac{\langle\alpha,\nu\rangle}{\langle\alpha,\nu\rangle + k_\alpha} \prod_{x\alpha<0} \frac{\langle\alpha,\nu\rangle - k_\alpha}{\langle\alpha,\nu\rangle + k_\alpha}.\end{aligned} \tag{5.1.7}$$

The equivalence of the two formulas can be easily seen by the substitution $x^{-1}\alpha \mapsto \alpha$ in the second product. Notice that the factor $(-1)^{|R|}\prod_{\alpha>0}\frac{\langle\alpha,\nu\rangle}{\langle\alpha,\nu\rangle+k_\alpha}$ is independent of x, so we may divide the form uniformly by it. The resulting normalized hermitian form has the property that

$$\langle \mathcal{R}_1 \otimes \mathbb{1}_\nu, \mathcal{R}_1 \otimes \mathbb{1}_\nu\rangle_\bullet = 1.$$

When ν is dominant, $k_\alpha + \langle\alpha,\nu\rangle > 0$, so the denominator does not vanish, and it is always positive (we have assumed $k_\alpha > 0$).

The arguments also imply that $\langle h_2, h_1\rangle_\bullet = \overline{\langle h_1, h_2\rangle_\bullet}$ for $h_1, h_2 \in \{R_x \otimes \mathbb{1}_\nu\}_{x\in W}$, so also in general. In conclusion, we have proved the following result.

Proposition 5.1.2. *The form*

$$\langle h_1, h_2\rangle_\bullet := \epsilon_A(t_{w_0} h_2^\bullet h_1 R_{w_0})(w_0\nu)$$

defines a $\bullet$-invariant hermitian form on $X(\nu)$ if and only if $\overline{\nu} = \nu$, i.e., $\nu \in V^\vee$.

The case of $\star$ follows by formal manipulations. Set

$$\langle h_1, h_2\rangle_\star = \epsilon_A\left(h_2^\star h_1 R_{w_0}\right)(w_0\nu). \tag{5.1.8}$$

The relation between the forms is

$$\begin{aligned}\langle h_1, h_2\rangle_\star &= \epsilon_A\left(h_2^\star h_1 R_{w_0}\right)(w_0\nu) = \epsilon_A\left(t_{w_0}\delta(h_2)^\bullet t_{w_0} h_1 R_{w_0}\right)(w_0\nu) \\ &= \langle t_{w_0} h_1, \delta(h_2)\rangle_\bullet.\end{aligned} \tag{5.1.9}$$

We also note the following formulas for the signatures.

Proposition 5.1.3. *Write $R_{w_0} = \sum_{w\in W} t_w a_w$.*

1. *The signature of $\langle\ ,\ \rangle_\bullet$ is given by the signature of the matrix $\left\{a_{x^{-1}yw_0}\right\}_{x,y\in W}$.*
2. *The signature of $\langle\ ,\ \rangle_\star$ is given by the signature of the matrix $\left\{a_{x^{-1}y}\right\}_{x,y\in W}$.*

Proof. Straightforward. □

Corollary 5.1.4. *For every $w \in W$,*

$$\epsilon_A\left(t_w R_{w_0}\right) = \epsilon_A\left(\delta(t_{w^{-1}})R_{w_0}\right).$$

Proof. The left hand side is

$$\epsilon_A\left(t_{w_0} t_{w_0} t_w R_{w_0}\right),$$

while the right hand side is

$$\epsilon_A\left(t_{w_0} t_{w^{-1}} t_{w_0} R_{w_0}\right)$$

Evaluating at $w_0\nu$, the left hand side is $\langle t_{w_0}, t_w\rangle_{\bullet,\nu}$ while the right hand side is $\langle t_w, t_{w_0}\rangle_{\bullet,\nu}$. The fact that the two are equal follows from the fact that $\langle\ ,\ \rangle_\bullet$ is symmetric for ν real. □

As a consequence of the relation (5.1.9) between $\bullet$ and $\star$ forms and Proposition 5.1.2, we have the following corollary.

Corollary 5.1.5. *The pairing*

$$\langle h_1, h_2\rangle_\star = \epsilon_A\,(h_2^\star h_1 R_{w_0})\,(w_0\nu)$$

defines a $\star$-invariant hermitian form on $X(\nu)$ if and only if $w_0\nu = -\overline{\nu}$.

6. Positive definite forms: spherical modules

The spherical $\star$-unitary dual of graded Hecke algebras with equal parameters is known by [Ba], [BM3], [BC3]. For Hecke algebras with unequal parameters, the irreducible $\star$-unitary modules that are both spherical and generic were determined in [BC2]. The Dirac inequality [BCT] is far from sufficient to determine the answer. In this section, we show that the Dirac inequality **is** sufficient to compute the spherical $\bullet$-unitary dual, at least in the case of the graded Hecke algebra with equal parameters. As a result, the answer, Theorem 6.3.3 is much simpler than the answer for the spherical $\star$-unitary dual in *loc. cit.* Theorem 6.3.3 complements the results in [Op2, sections 4 and 5].

6.1. **The Dirac operator.** We assume that the Hecke algebra $\mathbb{H}$ has equal parameters $k_\alpha = 1$.

We fix a W-invariant inner product $\langle\ ,\ \rangle$ on V. Let $O(V)$ denote the orthogonal group of V with respect to $\langle\ ,\ \rangle$. Then $W \subset O(V)$. Denote also by $\langle\ ,\ \rangle$ the dual inner product on $V^\vee$. If v is a vector in V or $V^\vee$, we denote $|v| := \langle v, v\rangle^{1/2}$.

Denote by $C(V)$ the Clifford algebra defined by $(V, \langle\ ,\ \rangle)$. Precisely, $C(V)$ is the associative algebra with unit generated by V with relations:

$$\omega^2 = -\langle\omega,\omega\rangle, \quad \omega\omega' + \omega'\omega = -2\langle\omega,\omega'\rangle. \tag{6.1.1}$$

Let $\mathsf{Pin}(V)$ be the Pin group, a double cover of $O(V)$ with projection map $p : \mathsf{Pin} \longrightarrow O(V)$, and let $\widetilde{W} = p^{-1}(W) \subset \mathsf{Pin}(V)$ be the pin double cover of W. See [BCT] for more details.

If $\dim V$ is even, the Clifford algebra $C(V)$ has a unique complex simple module (γ, S) of dimension $2^{\dim V/2}$. When $\dim V$ is odd, there are two simple inequivalent complex modules (γ_+, S^+), (γ_-, S^-) of dimension $2^{[\dim V/2]}$. Fix S to be one of these simple modules. The choice will not play a role in the present considerations. Endow S with a positive definite invariant Hermitian form $\langle\ ,\ \rangle_S$.

We call a representation $\widetilde{\sigma}$ of $\widetilde{W}$ genuine if it does not factor through W. For example, S is a genuine $\widetilde{W}$-representation.

Let $\{\omega_i : i = 1, n\}$ be an orthonormal basis of V with respect to $\langle\ ,\ \rangle$. Define ([BCT]) the Casimir element of $\mathbb{H}$:

$$\Omega = \sum_{i=1}^{n} \omega_i^2 \in \mathbb{H}.$$

It is easy to see that the element Ω is independent of the choice of bases and central in $\mathbb{H}$. Moreover, if (π, X) is an irreducible $\mathbb{H}$-module, then $\pi(\Omega)$ acts by the scalar $\langle \mathsf{cc}(X), \mathsf{cc}(X)\rangle$. Note that $\langle\ ,\ \rangle$ is extended linearly to $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$, and $\mathsf{cc}(X)$ stands for **any** representative of the set of weights.

For every $\omega \in V$, recall that we have defined

$$\widetilde{\omega} = \frac{1}{2}(\omega - \omega^\star) = \omega - \frac{1}{2}\sum_{\beta>0}(\beta,\omega)t_{s_\beta} \ \in \ \mathbb{H}. \tag{6.1.2}$$

It is immediate that

$$\widetilde{\omega}^\star = -\widetilde{\omega} \text{ and } \widetilde{\omega}^\bullet = \widetilde{\omega}. \tag{6.1.3}$$

The Dirac element ([BCT]) is defined as

$$\mathcal{D} = \sum_i \widetilde{\omega}_i \otimes \omega_i \in \mathbb{H} \otimes C(V).$$

For a finite dimensional module X of $\mathbb{H}$, define the Dirac operator $D : X \otimes S \to X \otimes S$ for X (and S) as given by the action of $\mathcal{D}$.

6.2. **Dirac inequality.** Let κ be one of the star operations $\star$ or $\bullet$.

Lemma 6.2.1. *The relations*

$$\Omega^\star = \Omega \textit{ and } \Omega^\bullet = \Omega \tag{6.2.1}$$

hold. Therefore, if X is κ-hermitian,

$$\begin{aligned} &\overline{\langle \mathsf{cc}(X), \mathsf{cc}(X)\rangle} = \langle \mathsf{cc}(X), \mathsf{cc}(X)\rangle, \ \textit{or, equivalently,} \\ &\langle \mathsf{cc}(X), \mathsf{cc}(X)\rangle = |\Re \mathsf{cc}(X)|^2 - |\Im \mathsf{cc}(X)|^2. \end{aligned} \tag{6.2.2}$$

Proof. For $\star$: $\Omega^\star = t_{w_0} \cdot (\overline{-w_0(\Omega)}) \cdot t_{w_0} = t_{w_0} \cdot \Omega \cdot t_{w_0} = \Omega$, where the last equality follows from the fact that $\Omega \in Z(\mathbb{H})$.

For $\bullet$: $\Omega^\bullet = \overline{\Omega} = \Omega$. □

Suppose X is a κ-hermitian $\mathbb{H}$-module with invariant form $(\ ,\ )_X$. Then $X \otimes S$ gets the Hermitian form $(x \otimes s, x' \otimes s')_{X\otimes S} = (x,x')_X \langle s, s'\rangle_S$. The operator D is self adjoint with respect to $(\ ,\ )_{X\otimes S}$ if $\kappa = \star$ and it is skew-adjoint if $\kappa = \bullet$:

$$D^* = D \text{ and } D^\bullet = -D. \tag{6.2.3}$$

Thus a κ-hermitian $\mathbb{H}$-module is κ-unitary only if for all $x \in X \otimes S$,

$$\begin{aligned} &(D^2x, x)_{X\otimes S} \geq 0, && \text{if } \kappa = \star, \text{ or} \\ &(D^2x, x)_{X\otimes S} \leq 0, && \text{if } \kappa = \bullet. \end{aligned} \tag{6.2.4}$$

We write $\Delta_{\widetilde{W}}$ for the diagonal embedding of $\mathbb{C}[\widetilde{W}]$ into $\mathbb{H} \otimes C(V)$ defined by extending $\Delta_{\widetilde{W}}(\widetilde{w}) = t_{p(\widetilde{w})} \otimes \widetilde{w}$ linearly.

Theorem 6.2.2 ([BCT, Theorem 3.5])**.** *The square of the Dirac element equals*

$$\mathcal{D}^2 = -\Omega \otimes 1 + \Delta_{\widetilde{W}}(\Omega_{\widetilde{W}}), \tag{6.2.5}$$

in $\mathbb{H} \otimes C(V)$, where

$$\Omega_{\widetilde{W}} = -\frac{1}{4}\sum_{\substack{\alpha>0,\beta>0\\ s_\alpha(\beta)<0}} |\alpha^\vee||\beta^\vee| \ \widetilde{s}_\alpha \widetilde{s}_\beta \in \mathbb{C}[\widetilde{W}]^{\widetilde{W}}. \tag{6.2.6}$$

Denote

$$\widetilde{\Sigma}(X) = \{\widetilde{\sigma} \in \mathsf{Irr}\ \widetilde{W} : \mathrm{Hom}_{\widetilde{W}}[\widetilde{\sigma}, X \otimes S] \neq 0\}. \tag{6.2.7}$$

Corollary 6.2.3 (Dirac Inequality)**.** *Let X be a κ-unitary $\mathbb{H}$-module.*

(1) *If $\kappa = \star$, then $|\Re \mathsf{cc}(X)|^2 - |\Im \mathsf{cc}(X)|^2 \leq \min\{\widetilde{\sigma}(\Omega_{\widetilde{W}}) : \widetilde{\sigma} \in \widetilde{\Sigma}(X)\}$.*

(2) *If* $\kappa = \bullet$, *then* $|\Re cc(X)|^2 - |\Im cc(X)|^2 \geq \max\{\widetilde\sigma(\Omega_{\widetilde W}) : \widetilde\sigma \in \widetilde\Sigma(X)\}$.

Proof. This is an immediate corollary of Theorem 6.2.2 and (6.2.4). □

6.3. **Spherical modules.** Let $X(\nu) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu$ be the spherical principal series, $\nu \in V_{\mathbb{C}}^\vee$ and let $\overline{X}(\nu)$ be the unique spherical subquotient, *i.e.* $\mathrm{Hom}_W[\mathsf{triv}, \overline{X}(\nu)] \neq 0$.

Lemma 6.3.1. *Suppose* $\overline{X}(\nu)$ *is* $\bullet$*-unitary. Then* $|\Re\nu|^2 \geq |\Im\nu|^2 + |\rho^\vee|^2$.

Proof. Since $\mathrm{Hom}_W[\mathsf{triv}, \overline{X}(\nu)] \neq 0$, we have $S \in \widetilde\Sigma(\overline{X}(\nu))$. It is known and easy to see that $S(\Omega_{\widetilde W}) = |\rho^\vee|^2$. The claim then follows from Corollary 6.2.3. □

The parameters $\nu \in V_{\mathbb{C}}^\vee$ for which the spherical principal series $X(\nu)$ becomes reducible are known, see [Ch] for the general case. This also follows from the Kazhdan-Lusztig classification, proved in the graded affine Hecke algebra case in [Lu2]. When the parameter ν is regular, the reducibility is a consequence of intertwining operator calculations and it goes back in the setting of p-adic groups to Casselman [Cas2]. In the equal parameters case, the result is that $X(\nu)$ is reducible if and only if:

$$(\alpha, \nu) = \pm 1, \text{ for some } \alpha \in R^+. \tag{6.3.1}$$

Suppose that $\nu \in V^\vee$ is dominant, i.e., $(\alpha, \nu) \geq 0$, for all $\alpha \in \Pi$. The reducibility hyperplanes $\alpha = 1$, $\alpha \in R^+$, define an arrangement of hyperplanes in the dominant Weyl chamber in $V^\vee$. One open region in the complement of this arrangement of hyperplanes is

$$\mathcal{C}_\infty = \{\nu \in V^\vee : (\alpha, \nu) > 1,\ \alpha \in \Pi\}. \tag{6.3.2}$$

Lemma 6.3.2 (see also [Op2, Theorem 4.1])**.** *If* $\nu \in \overline{\mathcal{C}}_\infty$, *then* $\overline{X}(\nu)$ *is* $\bullet$*-unitary.*

Proof. It is sufficient to prove that $X(\nu)$ is $\bullet$-unitary when $\nu \in \mathcal{C}_\infty$. If $\nu \in V^\vee$ is such that $(\alpha, \nu) \neq \pm 1$, then $X(\nu)$ is $\mathbb{A}$-semisimple with a basis of weight vectors given by $\mathcal{R}_x \otimes \mathbb{1}_\nu$, see Lemma 5.1.1. By (5.1.7), the $\bullet$-form in this basis is diagonal, and if we normalize the form so that $\langle \mathcal{R}_1 \otimes \mathbb{1}_\nu, \mathcal{R}_1 \otimes \mathbb{1}_\nu \rangle_\bullet = 1$, then

$$\langle \mathcal{R}_x \otimes \mathbb{1}_\nu, \mathcal{R}_x \otimes \mathbb{1}_\nu \rangle_\bullet = \prod_{x\nu<0} \frac{(\alpha, \nu) - 1}{(\alpha, \nu) + 1}. \tag{6.3.3}$$

Clearly, if $\nu \in \mathcal{C}_\infty$, then $\langle \mathcal{R}_x \otimes \mathbb{1}_\nu, \mathcal{R}_x \otimes \mathbb{1}_\nu \rangle_\bullet > 0$ for all $x \in W$. □

Theorem 6.3.3. *Suppose* ν *is dominant.*

(1) *If* $\Im\nu = 0$, *then* $\overline{X}(\nu)$ *is* $\bullet$*-unitary if and only if* $\nu \in \overline{\mathcal{C}}_\infty$.
(2) *If* $\Im\nu \neq 0$, *then* $\overline{X}(\nu)$ *is not* $\bullet$*-unitary.*

Proof. (1) Suppose first that $\Im\nu = 0$. By Lemma 6.3.2, $\overline{X}(\nu)$ is $\bullet$-unitary also for $\nu \in \overline{\mathcal{C}}_\infty$.

For the converse, notice that by Lemma 6.3.1, $|\nu| \geq |\rho^\vee|$ is a necessary condition for $\overline{X}(\nu)$ to be $\bullet$-unitary. Let $B(0, |\rho^\vee|) \subset V^\vee$ be the open ball of radius $|\rho^\vee|$ centered at the origin. Let $\mathcal{C}$ be an arbitrary cell in the arrangement of hyperplanes $\alpha = 1$ in the dominant Weyl chamber of $V^\vee$. Since the signature of the hermitian form of $\overline{X}(\nu)$ is the same for all $\nu \in \mathcal{C}$ (see [BC2, Theorem 2.4]), a necessary condition for $\bullet$-unitarity in $\mathcal{C}$ is that

$$\overline{\mathcal{C}} \cap B(0, |\rho^\vee|) = \emptyset. \tag{6.3.4}$$

We claim that this condition only holds when $\mathcal{C} \subset \overline{\mathcal{C}}_\infty$.

The cell $\mathcal{C}$ is characterized by the sets:

$J_0(\mathcal{C}) = \{\alpha \in R^+ : (\alpha, \nu) = 1, \forall \nu \in \mathcal{C}\}$,

$J_+(\mathcal{C}) = \{\alpha \in R^+ : (\alpha, \nu) > 1, \forall \nu \in \mathcal{C}\}$, $J_-(\mathcal{C}) = \{\alpha \in R^+ : (\alpha, \nu) < 1, \forall \nu \in \mathcal{C}\}$.

Suppose $\mathcal{C} \not\subset \overline{\mathcal{C}}_\infty$. Then $J_-(\mathcal{C}) \cap \Pi \neq \emptyset$. List the simple roots as $\{\alpha_1, \dots, \alpha_m, \dots, \alpha_n\}$, $m < n$, such that $\{\alpha_1, \dots, \alpha_m\} \subset J_0(\mathcal{C}) \cup J_+(\mathcal{C})$ and $\{\alpha_{m+1}, \dots, \alpha_n\} \subset J_-(\mathcal{C})$. Let $\{\omega_1^\vee, \dots, \omega_n^\vee\} \subset V^\vee$ be the fundamental coweights, and write $\nu = \sum_{i=1}^n c_i \omega_i^\vee$. Then

$$c_i \geq 1,\ i = 1, m, \quad c_j < 1,\ j = m+1, n.$$

Deform $c_i \to 1$, $i = 1, m$ to get the point $\nu' = \sum_{i=1}^n c_i' \omega_j^\vee$, where $c_i' = 1$ if $i = 1, m$ and $c_i' = c_i$ if $i = m+1, n$. Notice that if $(\beta, \nu) < 1$, then $(\beta, \nu') < 1$ also, and if $(\gamma, \nu) \geq 1$ then $(\gamma, \nu') \geq 1$ again. This means that $\nu' \in \overline{\mathcal{C}}$. But now:

$$\langle \nu', \nu' \rangle = \sum_{i,j} c_i' c_j' \langle \omega_i^\vee, \omega_j^\vee \rangle < \sum_{i,j} \langle \omega_i^\vee, \omega_j^\vee \rangle = \langle \rho^\vee, \rho^\vee \rangle, \tag{6.3.5}$$

and this means that $\nu' \in B(0, |\rho^\vee|)$, contradiction. We have used here implicitly that $\langle \omega_i^\vee, \omega_j^\vee \rangle \geq 0$. Thus claim (1) is proven.

For claim (2), fix $b \in \sqrt{-1} V^\vee \setminus \{0\}$, and set $\Pi_b = \{\alpha \in \Pi : (\alpha, b) = 0\} \subsetneq \Pi$. Let R_b be the root subsystem defined by Π_b with positive roots R_b^+. Let $\nu \in V^\vee$ dominant and we look at the principal series $X(\nu + b)$.

This is reducible if and only if there exists $\beta \in R_b^+$ such that $(\beta, \nu) = 1$. Repeating the argument above with R_b^+ in place of R^+, we find that every cell $\mathcal{C}$ contains in its closure a point ν such that $|\nu| \leq |\rho_b^\vee|$. But now, the Dirac inequality in Lemma 6.3.1 gives the necessary condition

$$|\nu|^2 \geq |b|^2 + |\rho^\vee|^2 > |\rho_b^\vee|^2, \tag{6.3.6}$$

and this is not satisfied. $\square$

6.4. **Dirac cohomology.** Let X be a finite dimensional module of $\mathbb{H}$. The Dirac cohomology of X (with respect to the fixed spin module S) is

$$H^D(X) = \ker D / \ker D \cap \operatorname{Im} D. \tag{6.4.1}$$

We say that X has nonzero Dirac cohomology if $H_D(X) \neq 0$ for a choice of spin module S.

Recall that when X is κ-unitary, then the operator D is self-adjoint when $\kappa = \star$ and skew-adjoint when $\kappa = \bullet$. This implies that if X is κ-unitary, then

$$H^D(X) = \ker D. \tag{6.4.2}$$

Theorem 6.3.3 says, in particular, that every irreducible subquotient of $X(\rho^\vee)$ is $\bullet$-unitary, so for all these subquotients, equation (6.4.2) applies.

The classification of irreducible subquotients of $X(\rho^\vee)$ is well known. In the setting of p-adic groups, it is due to Casselman [Cas3]. Each standard module with central character $\rho^\vee$ is of the form

$$X_M = \mathbb{H} \otimes_{\mathbb{H}_M} (\mathsf{St} \otimes \mathbb{C}_{\nu_M}),\ \text{where } \nu_M = \rho^\vee - \rho_M^\vee, \tag{6.4.3}$$

for $M \subset \Pi$. Let $\overline{X}_M$ be the Langlands quotient of X_M. Here $\mathbb{H}_M$ denotes the graded Hecke algebra defined by the root system $(V, R_M, V^\vee, R_M^\vee)$, where R_M is the subset of roots spanned by M. (We embed $\mathbb{H}_M$ as a subalgebra of $\mathbb{H}$ in the natural way.)

Thus we have a one-to-one correspondence between irreducible modules with central character $\rho^\vee$ and subsets of simple roots Π. In particular, there are 2^n, $n = |\Pi|$, distinct simple modules with central character $\rho^\vee$. Moreover, the following known character formula holds:

$$\overline{X}_M = \sum_{M\subseteq J\subseteq \Pi} (-1)^{|J|-|M|} X_J. \tag{6.4.4}$$

For example, this formula follows via the Kazhdan-Lusztig classification (and conjecture) [Lu2] since all the $G(\rho^\vee)$-orbits on $\mathfrak{g}_1(\rho^\vee)$ have smooth closure.

Proposition 6.4.1. *Suppose $V_{\mathbb{C}}^W = 0$, i.e, the root system is semisimple. For every $M \subset \Pi$, we have* $\dim \operatorname{Hom}_W[\overline{X}_M, \bigwedge^* V_{\mathbb{C}}] = 1$.

Proof. The character formula (6.4.4) implies

$$\begin{aligned}\operatorname{Hom}_W[\overline{X}_M, \bigwedge^* V] &= \sum_{J\supseteq M} (-1)^{|J|-|M|} \operatorname{Hom}_W[X_J, \bigwedge^* V_{\mathbb{C}}] \\ &= \sum_{J\supseteq M} (-1)^{|J|-|M|} \operatorname{Hom}_{W_J}[\mathsf{sgn}, \bigwedge^* (V_{\mathbb{C}})|_{W_J}],\end{aligned} \tag{6.4.5}$$

by Frobenius reciprocity. Let $V_{J,\mathbb{C}}$ be the $\mathbb{C}$-span of the roots in J, and $V_{J,\mathbb{C}}^\perp$ the orthogonal complement, so that $V = V_{J,\mathbb{C}} \oplus V_{J,\mathbb{C}}^\perp$. Notice that W_J acts by the reflection representation on $V_{J,\mathbb{C}}$ and trivially on $V_{J_{\mathbb{C}}}^\perp$. Since

$$\bigwedge^k (V_{\mathbb{C}})|_{W_J} = \bigoplus_{i=0}^k \bigwedge^i V_{J,\mathbb{C}} \otimes \bigwedge^{k-i} V_{J,\mathbb{C}}^\perp,$$

we see that

$$\dim \operatorname{Hom}_{W_J}[\mathsf{sgn}, \bigwedge^k (V_{\mathbb{C}})|_{W_J}] = \begin{cases} \dim \bigwedge^{k-j} V_{J,\mathbb{C}}^\perp, & k \geq |J|, \\ 0, & k < |J|. \end{cases} \tag{6.4.6}$$

This means that $\dim \operatorname{Hom}_{W_J}[\mathsf{sgn}, \bigwedge^*(V_{\mathbb{C}})|_{W_J}] = \dim \bigwedge^* V_{J,\mathbb{C}}^\perp = 2^{|\Pi|-|J|}$, and therefore

$$\operatorname{Hom}_W[\overline{X}_M, \bigwedge^* V_{\mathbb{C}}] = \sum_{M\subseteq J\subseteq \Pi} (-1)^{|J|-|M|} 2^{|\Pi|-|J|} = 1. \tag{6.4.7}$$

$\square$

Remark 6.4.2.

(1) An alternative proof of Proposition 6.4.1 is as follows. Use

$$\bigwedge^* V_{\mathbb{C}} = \begin{cases} S \otimes S, & \text{if } \dim V \text{ is even}, \\ \frac{1}{2}(S^+ + S^-) \otimes (S^+ + S^-), & \text{if } \dim V \text{ is odd}, \end{cases}$$

and rewrite in equation (6.4.5),

$$\operatorname{Hom}_W[X_J, \bigwedge^* V_{\mathbb{C}}] = a \operatorname{Hom}_{\widetilde{W}}[X_J \otimes \mathcal{S}, \mathcal{S}],$$

where $\mathcal{S} = S$, $a = 1$, if $\dim V$ is even, and $\mathcal{S} = S^+ + S^-$, $a = \frac{1}{2}$, if $\dim V$ is odd. Then we can apply [BC5, Lemma 2.6.2] which, in particular, gives the dimension of this space, and we arrive at formula (6.4.7).

(2) Notice that Proposition 6.4.1 says that every simple $\mathbb{H}$-module $\overline{X}_M$ at $\rho^\vee$ contains one and only one W-type that appears in $\bigwedge^* V_{\mathbb{C}}$. It is worth recalling that when the root system is simple, every $\bigwedge^k V_{\mathbb{C}}$ is in fact irreducible as a W-representation and that any two distinct exterior powers are nonisomorphic, see [GP, Theorem 5.1.4].

(3) Since $\dim \mathrm{Hom}_W[X(\rho^\vee), \bigwedge^* V_{\mathbb{C}}] = \dim \mathrm{Hom}_W[\mathbb{C}[W], \bigwedge^* V_{\mathbb{C}}] = \dim \bigwedge^* V_{\mathbb{C}} = 2^{|\Pi|}$ and this is also the number of distinct simple $\mathbb{H}$-modules with central character $\rho^\vee$, Proposition 6.4.1 implies that each irreducible constituent σ of $\bigwedge^* V_{\mathbb{C}}$ occurs in exactly $\dim \sigma$ different such simple $\mathbb{H}$-modules.

Example 6.4.3. In the case of the Hecke algebra of type A_{n-1}, one can determine exactly which constituent of $\bigwedge^* V_{\mathbb{C}}$, where $V_{\mathbb{C}} \cong \mathbb{C}^{n-1}$, appears in each simple $\mathbb{H}$-module with central character $\rho^\vee$. Consider a composition of n, i.e., a k-tuple $(n_1, n_2, \ldots, n_k)$ where $n_i > 0$, $\sum n_i = n$ and let $\overline{X}(n_1, n_2, \ldots, n_k)$ be the simple module at $\rho^\vee$ corresponding to the subset of the simple roots $\mathcal{S}(n_1, n_2, \ldots, n_k) := \Pi \setminus \{\alpha_{n_1}, \alpha_{n_1+n_2}, \ldots\}$. Notice that the standard module $X(n_1, n_2, \ldots, n_k)$ contains the hook S_n-representation $(k, 1, 1, \ldots, 1) = \bigwedge^{n-k} V_{\mathbb{C}}$ with multiplicity 1. Moreover, if $\mathcal{S}(n_1, n_2, \ldots, n_k) \subsetneq \mathcal{S}(n'_1, n'_2, \ldots, n'_{k'})$, then $k' < k$, and $(k, 1, 1, \ldots, 1)$ does not appear in the standard module $X(n'_1, n'_2, \ldots, n'_{k'})$. Therefore, by (6.4.4), we see that $\bigwedge^{n-k} V_{\mathbb{C}}$ appears with multiplicity 1 in $\overline{X}(n_1, n_2, \ldots, n_k)$.

Corollary 6.4.4. *Suppose the root system is semisimple. For every simple $\mathbb{H}$-module $\overline{X}$ with central character $\rho^\vee$,*

$$H^D(\overline{X}) = S,$$

for an appropriate choice of spin module S.

Proof. Let $\overline{X}_M$ be a simple $\mathbb{H}$-module at $\rho^\vee$, $M \subset \Pi$. As remarked above, $\overline{X}_M$ is $\bullet$-unitary and therefore $H^D(X) = \ker D$. Since $S(\Omega_{\widetilde{W}}) = \langle \rho^\vee, \rho^\vee \rangle = \langle \mathsf{cc}(\overline{X}_M), \mathsf{cc}(\overline{X}_M) \rangle$, a known argument, e.g. [BCT, Proposition 5.7], says that $\ker D$ is nonzero if (in fact, if and only if) S occurs in $\overline{X}_M \otimes S$. But $\mathrm{Hom}_{\widetilde{W}}[\overline{X}_M \otimes S, S] = \mathrm{Hom}_W[\overline{X}_M, S \otimes S^*]$, and

$$\begin{aligned} &S \otimes S^* = \bigwedge^* V_{\mathbb{C}}, \text{ when } \dim V_{\mathbb{C}} \text{ is even,} \\ &(S^+ + S^-) \otimes (S^+ + S^-)^* = 2\bigwedge^* V_{\mathbb{C}}, \text{ when } \dim V_{\mathbb{C}} \text{ is odd.} \end{aligned} \tag{6.4.8}$$

Then Proposition 6.4.1, implies that $\dim \mathrm{Hom}_{\widetilde{W}}[\overline{X}_M \otimes S, S] = 1$ for some choice of S.

□

References

[ALTV] J. Adams, M. van Leeuwen, P. Trapa, D. Vogan, *Unitary representations of real reductive groups*, preprint, `arXiv:1212.2192`.

[Ba] D. Barbasch, *The unitary spherical spectrum for split classical groups*, J. Inst. Math. Jussieu **9** (2010), no. 2, 265–356.

[BC1] D. Barbasch, D. Ciubotaru, *Unitary equivalences for reductive p-adic groups*, Amer. Jour. Math. **135** (2013), no. 6, 1633–1674.

[BC2] D. Barbasch, D. Ciubotaru, *Spherical unitary principal series*, Pure Appl. Math. Q. **1** (2005), no. 4, Special Issue: In memory of Armand Borel. Part 3, 755–789.

[BC3] D. Barbasch, D. Ciubotaru, *Whittaker unitary dual of affine graded Hecke algebras of type E*, Compos. Math. **145** (2009), no. 6, 1563–1616.

[BC4] D. Barbasch, D. Ciubotaru, *Hermitian forms for affine Hecke algebras*, preprint, `arXiv:1312.3316v2`.

[BC5] D. Barbasch, D. Ciubotaru, *Unitary Hecke algebra modules with nonzero Dirac cohomology*, Symmetry in Representation Theory and Its Applications: In honor of Nolan Wallach, Progr. Math. Birkhäuser (2014), 1–20.

[BCT] D. Barbasch, D. Ciubotaru, P. Trapa, *Dirac cohomology for graded affine Hecke algebras*, Acta Math. **209** (2012), no. 2, 197–227.

[BM1] D. Barbasch, A. Moy, *A unitarity criterion for p-adic groups*, Invent. Math. **98** (1989), 19–38.

[BM2] D. Barbasch, A. Moy, *Reduction to real infinitesimal character in affine Hecke algebras*, J. Amer. Math. Soc. **6**(3), 1993, 611–635.

[BM3] D. Barbasch, A. Moy, *Unitary spherical spectrum for p-adic classical groups*, Acta Appl. Math. **44**, 1996, 1–37.

[BM4] D. Barbasch, A. Moy, *Classification of one K-type representations*, Trans. Amer. Math. Soc. **351**, no. 10, 4245–4261.

[Be] J. Bernstein, *Representations of p-adic groups*, lecture notes, Harvard University, 1992.

[BeMi] R. Bezrukavnikov, I. Mirković, *Representations of semi-simple Lie algebras in prime characteristic and noncommutative Springer resolution* (with an Appendix by Eric Sommers), Annals Math. **178** (2013), no. 3, 835–919.

[Cas1] W. Casselman, *Introduction to the theory of admissible representations of p-adic groups*, unpublished monograph.

[Cas2] W. Casselman, *The unramified principal series of p-adic groups. I. The spherical function*, Compositio Math. **40** (1980), no. 3, 387–406.

[Cas3] W. Casselman, *A new nonunitarity argument for p-adic representations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **28** (1981), 907–928.

[Ch] I. Cherednik, *A unification of Knizhnik-Zamolodchikov equations and Dunkl operators via affine Hecke algebras*, Invent. Math. **106** (1991), 411–432.

[CH] D. Ciubotaru, X. He, *Green polynomials of Weyl groups, elliptic pairings, and the extended Dirac index*, `arXiv:1303.6806v2` (2014).

[CM] D. Ciubotaru, A. Moy, *Dirac cohomology of one-W-type representations*, Proc. Amer. Math. Soc. **143** (2015), no. 3, 1001–1013.

[EM] S. Evens, I. Mirković, *Fourier transform and the Iwahori-Matsumoto involution*, Duke Math. J. **86** (1997), no. 3, 435–464.

[Ev] S. Evens, *The Langlands classification for graded Hecke algebras*, Proc. Amer. Math. Soc. **124** (1996), no. 4, 1285–1290.

[GP] M. Geck, G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs. New Series, vol. 21, The Clarendon Press Oxford University Press, New York, 2000.

[He] V. Heiermann, *Opérateurs d'entrelacement et algèbres de Hecke avec paramètres d'un groupe réductif p-adique: le cas des groupes classiques*, Selecta Math. (N.S.) **17** (2011), no. 3, 713–756.

[KL] D. Kazhdan, G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87**, 1987, 153–215.

[Lu1] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2**, 1989, 599–635.

[Lu2] G. Lusztig, *Cuspidal local systems and graded algebras II*, Representations of groups (Banff, AB, 1994), Amer. Math. Soc., Providence, 1995, 217–275.

[Lu3] G. Lusztig, *Classification of unipotent representations of simple p-adic groups,* Internat. Math. Res. Notices (1995), no. **11**, 517–589.

[Op1] E.M. Opdam, *On the spectral decomposition of affine Hecke algebras*, J. Math. Jussieu **3** (2004), no. 4, 531–648.

[Op2] E.M. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. **175** (1995), no. 1, 75–121.

[Sp] T.A. Springer, *Reductive groups*, Proc. Symp. Pure Math. vol. 33 (1979), part 1, 3–27.

[SV] W. Schmid, K. Vilonen, *Hodge theory and unitary representations of reductive Lie groups*, Frontiers of Mathematical Sciences, International Press 2011, 397–420.

[Ti] J. Tits, *Reductive groups over local fields*, Proceedings of Symposia in Pure Math., vol. 33 (1979), 29–69.

[Vo] D.A. Vogan, Jr., *Unitarizability of certain series of representations*, Annals Math., vol. **120**, no. 1 (1984), 141–187.

(D. Barbasch) Dept. of Mathematics, Cornell University, Ithaca, NY 14850
E-mail address: barbasch@math.cornell.edu

(D. Ciubotaru) Mathematical Institute, University of Oxford, Andrew Wiles Building, Oxford, OX2 6GG, UK
E-mail address: dan.ciubotaru@maths.ox.ac.uk